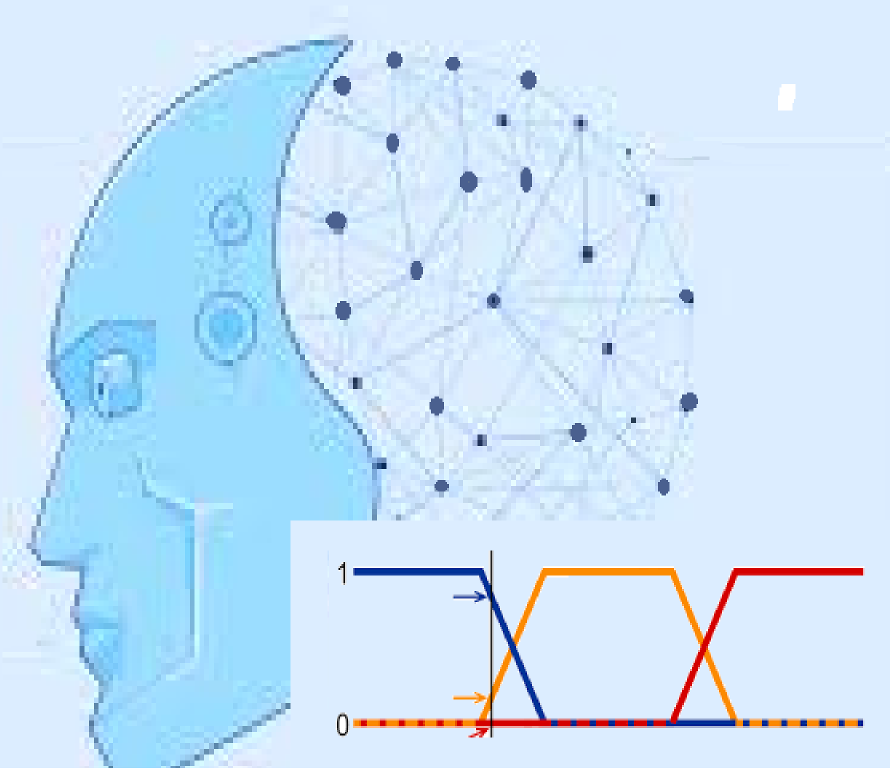
Soft Computing



FUZZY LOGIC

MODULE 3

Module - 3 (Fuzzy Logic & Defuzzification)

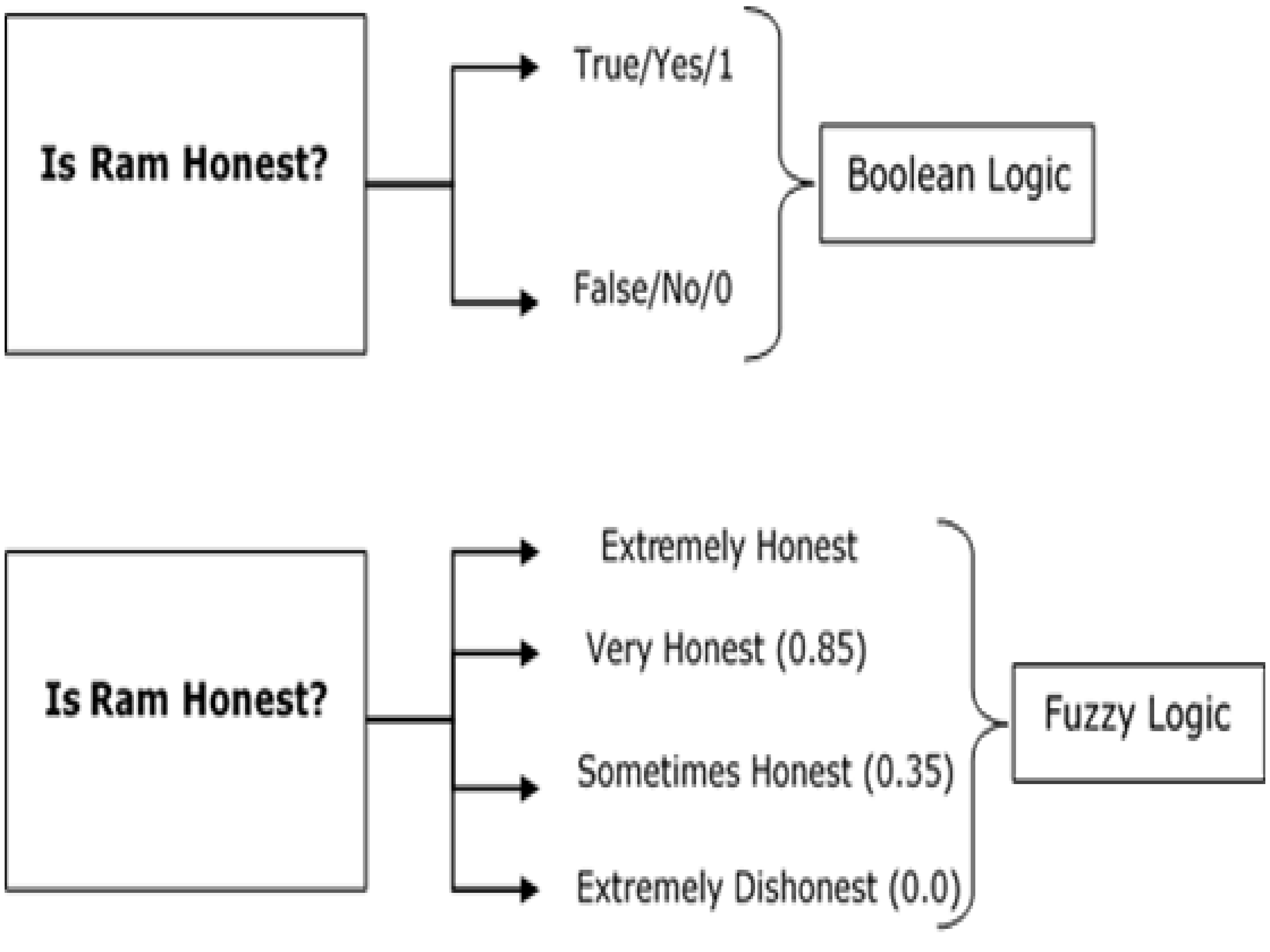
Fuzzy sets – properties, operations on fuzzy set. Fuzzy membership functions, Methods of membership value assignments – intuition, inference, Rank Ordering. Fuzzy relations– operations on fuzzy relation. Fuzzy Propositions. Fuzzy implications. Defuzzification– Lamda cuts, Defuzzification methods.

CRISP SETS

- Either an element belongs to the set or does not
- Everything is either True or False
- No uncertainty is allowed
- An item is
  - either entirely with in the set or
  - entirely not in the set

EXAMPLES

- For the set of integers, either an integer is even or it is not (it is odd)
- Either you are in Kerala or not (outside Kerala)
  - Crossing boarder
- For black and white photographs, a pixel is either black or not (white)


Is Ram Honest?
True/Yes/1
False/No/0
Boolean Logic
Is Ram Honest?
Extremely Honest
Very Honest (0.85)
Sometimes Honest (0.35)
Extremely Dishonest (0.0)
Fuzzy Logic

FUZZY LOGIC REPRESENTATION

- For every problem must represent in terms of fuzzy sets.
- What are fuzzy sets?

Slowest

[0.0 – 0.25]

Slow

[0.25 – 0.50]

Fast

[0.50 – 0.75]

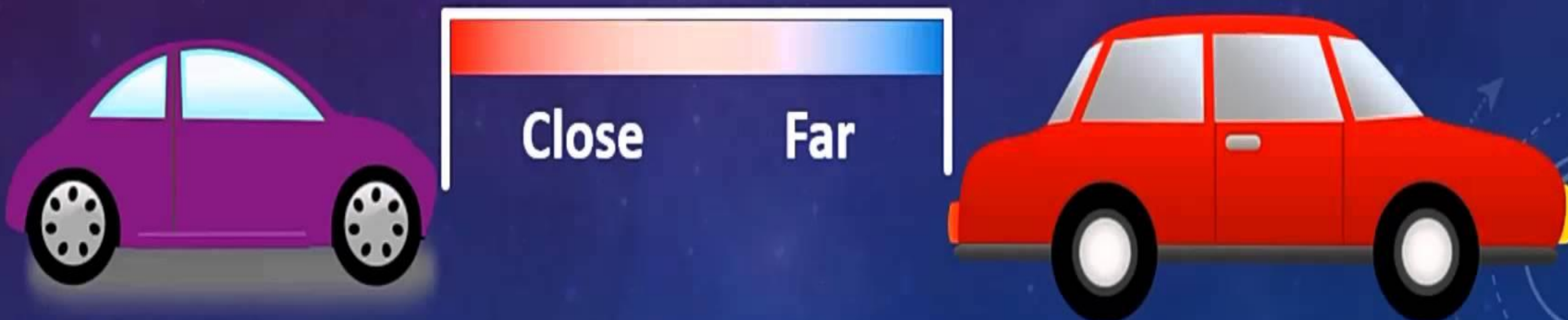
Fastest

[0.75 – 1.00]

WHY IS IT USEFUL?

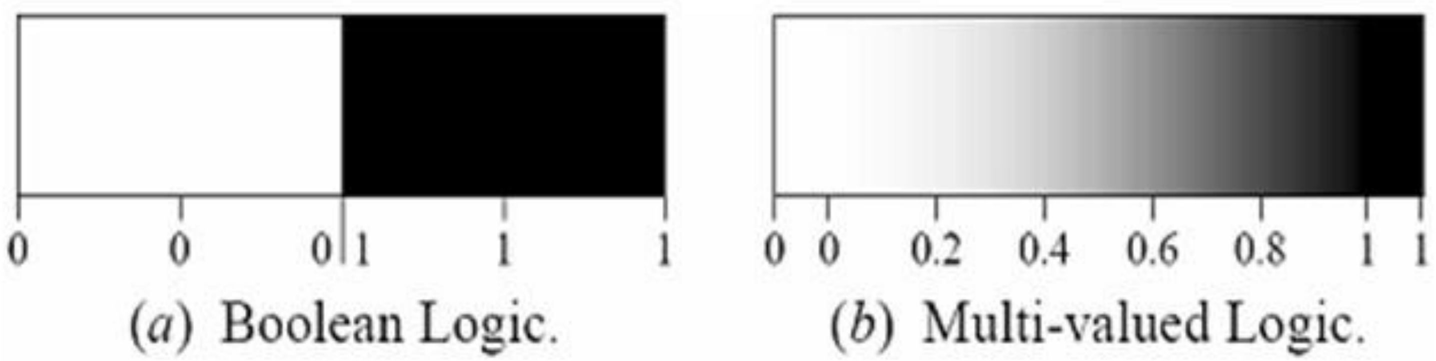
Automatic Braking System

Fuzzy Logic



Is car close? : 0-1 (Range of No to Yes)
Brakes : 0-1 (Range of Off to On)

- Boolean logic uses sharp distinctions.
- Fuzzy logic reflects how people think.



- Fuzzy logic is a set of mathematical principles for knowledge representation and reasoning based on degrees of membership.

REPRESENTATION

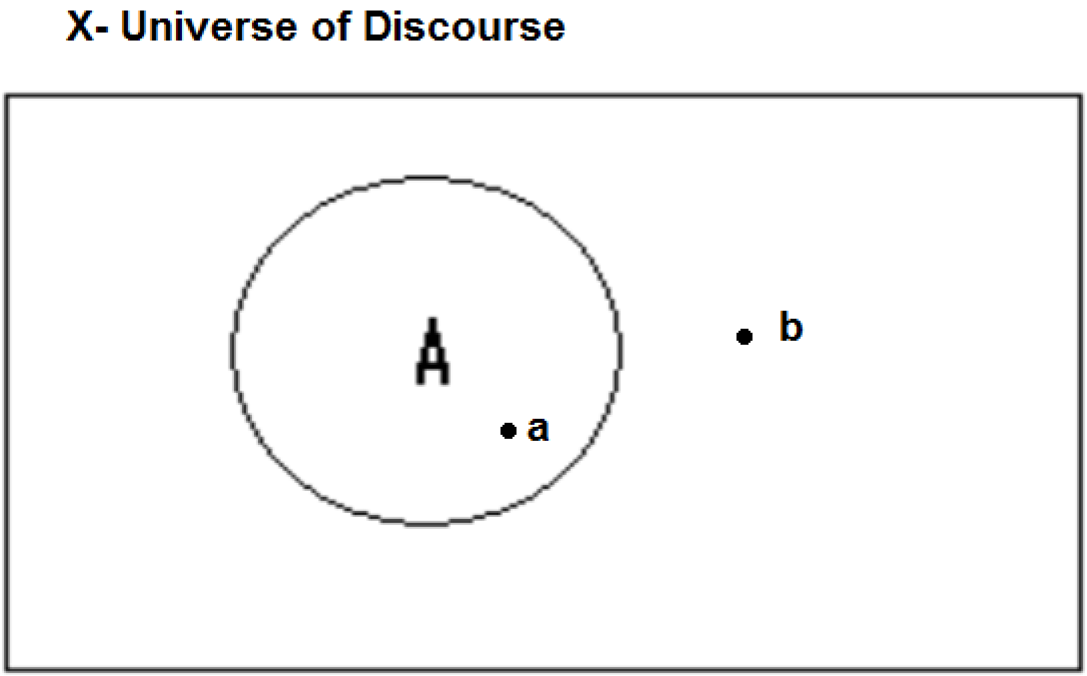
- As Lists – by enumerating all the elements
  - Examples- A = {apples, oranges, mangoes}
  - $A = \{2, 4, 6, 8, 10 \ldots\}$
- As Formulas:
  - Examples- $A = \{x \mid x$ is an even natural number$\}$
  - $A = \{x \mid x = 2n$, n is a natural number$\}$

CHARACTERISTIC FUNCTION

- Example

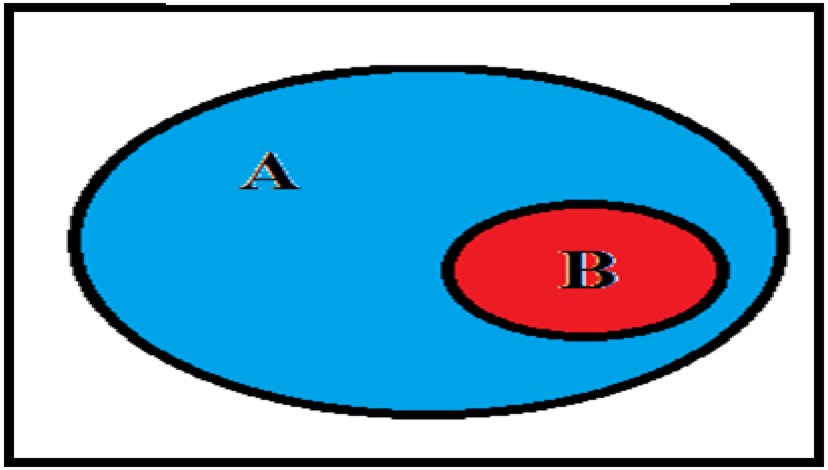
$$\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

VENN DIAGRAMS



SUBSET

- For sets A and B,
- B is a subset of A if B is contained in A or is equivalent to A - $B \subseteq A$
- Proper subset if B is completely contained in A
- $B \subset A$

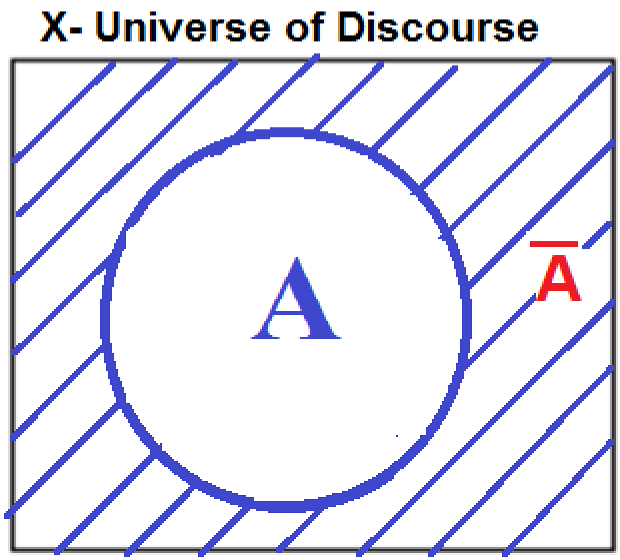


NULL SET

- Set with no elements
- Denoted by $\emptyset$
- The set of all possible subsets of A is called power set of A
- P(A)=$\{x|x \subseteq A\}$

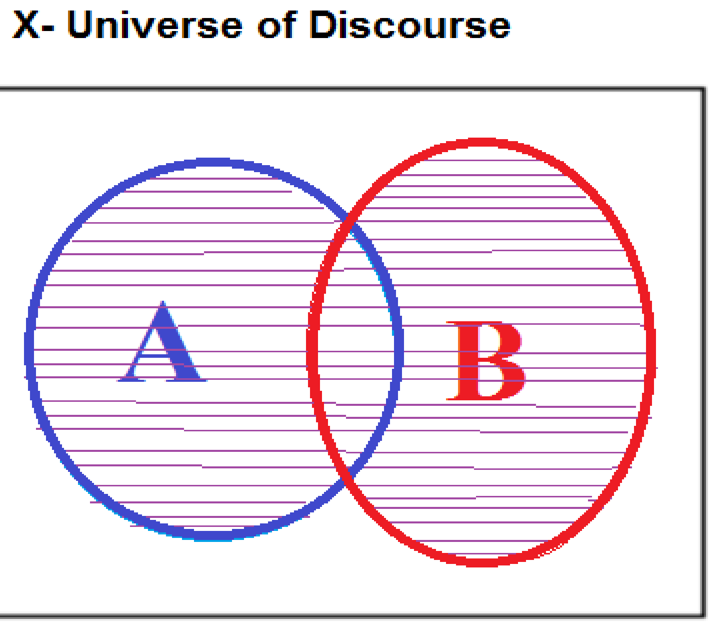
Operations On Sets-Complement

- Each element either in A or not in A- Complement $\bar{A}=\{x|x \notin A, x \in X\}$



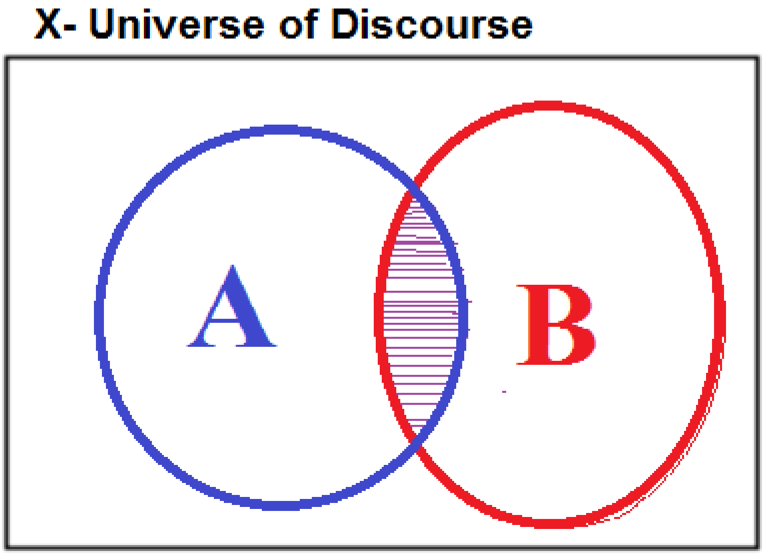
OPERATIONS ON SETS- UNION

- All those elements that belong to either A or B
- $A \cup B \ = \{x | x \in A or\ x \ \in B\}$



Operations On Sets- Intersection

- All those elements that belong to both A and B
- $A \cap B = \{x | x \in A and\ x \in B\}$



Fundamental Properties of Crisp Sets

- Involution $\overline{\overline{A}} = A$
- Commutativity $A \cup B = B \cup A$

  $A \cap B = B \cap A$
- Associativity

  $(A \cup B) \cup C = A \cup (B \cup C)$

  $(A \cap B) \cap C = A \cap (B \cap C)$

Fundamental Properties of Crisp Sets Contd..

- Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Fundamental Properties of Crisp Sets Contd..

- Idempotency

$$A \cup A = A$$

$$A \cap A = A$$

- Identity

$$A \cup \varnothing = A$$

$$A \cap X = A$$

FUNDAMENTAL PROPERTIES OF CRISP SETS CONTD..

- Law of contradiction

$$\mathbf{A} \cap \overline{\mathbf{A}} = \varnothing$$

- Law of excluded middle

$$\mathbf{A} \cup \overline{\mathbf{A}} = \mathbf{X}$$

DEMORGAN'S LAWS

- DeMorgan's laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

FUZZY SETS

- Introduced by Lotfi A Zadeh in 1960's
- Used to represent sets where boundary of information is unclear
- To account for concepts used in human reasoning which are vague and imprecise
- Elements can belongs to the set or not
- Strength of membership/ Degree of membership is associated

EXAMPLE

- Fuzzy set is very convenient method for representing some form of uncertainty
- For example: the weather today
  - Sunny: If we define any cloud cover of 25% or less is sunny
  - This means that a cloud cover of 26% is not sunny?
  - Vagueness should be introduced

DIFFERENCE

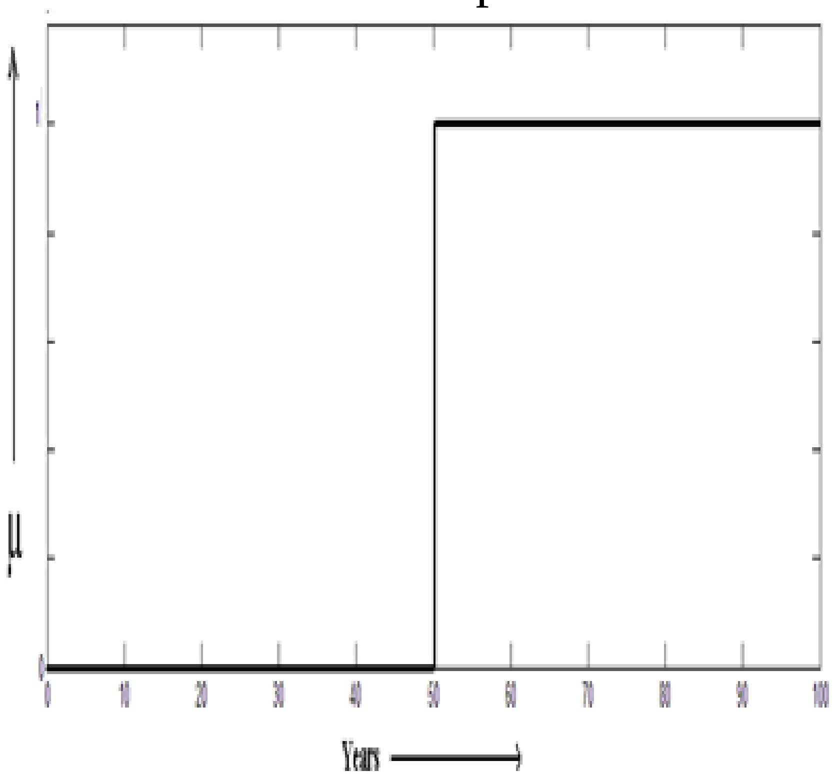
- Crisp Sets-Only two values possible
- Membership of element ‘x’ in set A is described by a characteristic function $\mu_A(x)$ which can be either 0 or 1
- Fuzzy sets - Extends using partial membership
- A fuzzy set A on a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the real interval [0, 1]

EXAMPLE

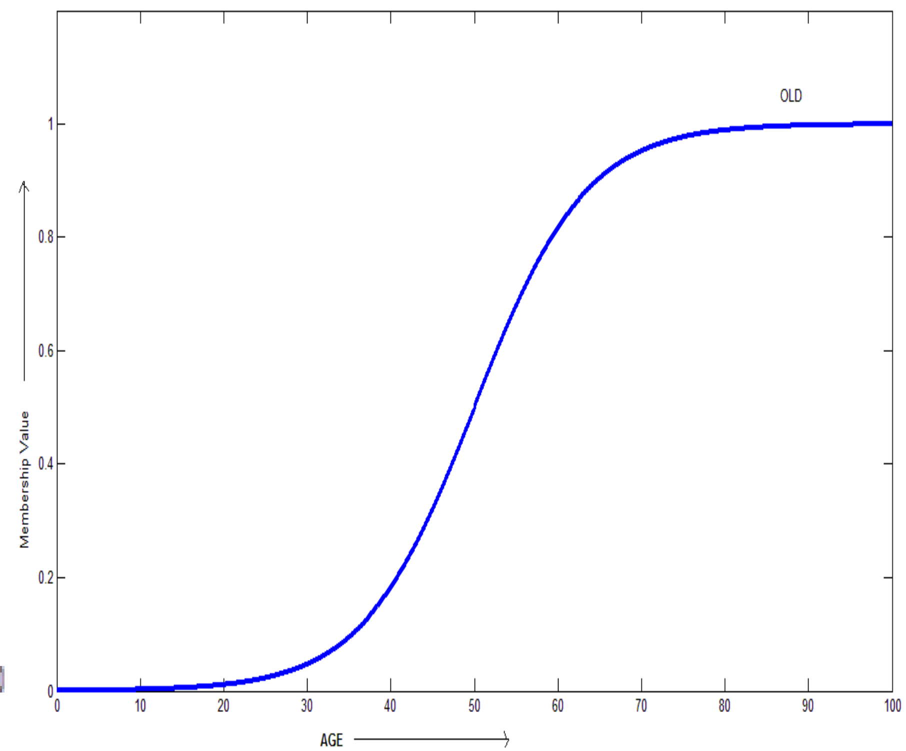
- Whether Mary is old?
- Mary is a member of the set of old people
- If Mary's age is 75, what is the strength of that belief in 0 to 1 range?
- Mary's degree of membership within the set of old people = 0.95
- Represented as membership function $\mu_{Old}(Mary) = 0.95$

GRAPHICAL REPRESENTATION

Crisp Set



Fuzzy Set



Membership Function for OLD

SET REPRESENTATION

- Universe of Discourse- X

FUZZY SETS-REPRESENTATION

- Let A isdefined on a finite universal set X where $x_1, x_2, \ldots, x_n$ denote elements of A, then A can be described as

$$A=\left\{\frac{\mu_A(x_1)}{x_1}+\frac{\mu_A(x_2)}{x_2}+\cdots\right\}=\left\{\sum_i \frac{\mu_A(xi)}{x_i}\right\}$$

Can be represented as $\{(x_1, \mu_A(x_1)),(x_2, \mu_A(x_2))\}$

$$A=\left\{\frac{0.9}{5}+\frac{0}{10}+\frac{0.8}{15}+\frac{1.0}{20}\right\}$$

FUZZY SETS-REPRESENTATION

- When the universe X is continuous and infinite the fuzzy set A can be described as

$$\left\{ \int \frac{\mu_A(x_1)}{x_1} \right\}$$

EXAMPLE ROOMS IN A HOUSE

- Suitability of a house with n rooms(n=1..6) for a 3 member family can be represented as a fuzzy set

$$A=\left\{\frac{0.2}{1}+\frac{0.6}{2}+\frac{0.8}{3}+\frac{1.0}{4}+\frac{0.7}{5}+\frac{0.2}{6}\right\}$$

- Another method of representation is

$$A=\{(1,0.2),(2,0.6),(3,0.8),(4,1.0),(5,0.7),(6,0.2)\}$$

FAMILY DATASET EXAMPLE

Family Member	Age	Gender	Senior Person
Grand-pa	72	M	.95
Grand-ma	70	F	.92
Dad	42	M	.5
Mom	37	F	.4
Daughter	13	F	0
Son	10	M	0
Aunty	47	F	.6

- Fuzzy Set Senior _Person ?

$$A=\left\{\frac{0.95}{\text{Grand−pa}}+\frac{0.92}{\text{Grand−ma}}+\frac{0.5}{\text{Dad}}+\frac{0.4}{\text{Mom}}+\frac{0}{\text{Daughter}}+\frac{0}{\text{Son}}+\frac{0.6}{\text{Aunty}}\right\}$$

OPERATIONS: FUZZY SETS- SUBSET

- Given two fuzzy set A, B defined on the Universe of Discourse X, then *A* is a subset of *B* denoted by $A \subseteq B$
- Iff $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$

FUZZY COMPLEMENT

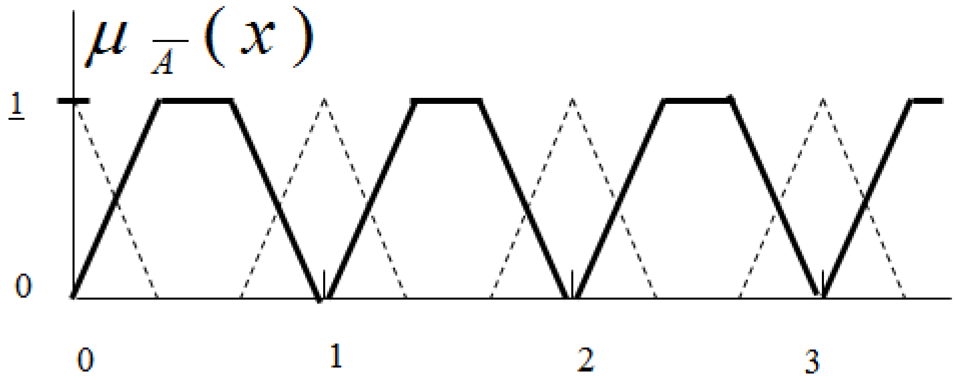
- For a fuzzy set A, $\bar{A}$ denotes the fuzzy complement of A
- Membership function for fuzzy complement is

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

EXAMPLE COMPLEMENT

- Complement *of A* ={ $x \mid x$ is <u>not</u> *near* an integer}

$$\boxed{\mu_{\bar{A}}(x) = 1 - \mu_A(x)}$$



NOT SMALL

- Find not small

$$\text{Small} = \left\{ \frac{1}{1} + \frac{0.64}{2} + \frac{0.36}{3} + \frac{0.16}{4} + \frac{0.04}{5} \right\}$$

$$\text{Not Small} = 1 - \text{Small}$$
$$= \left\{ \frac{0}{1} + \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5} \right\}$$

NOT LARGE

- Find not Large

$$\text{Large} = \left\{ \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} \right\}$$

$$= \left\{ \frac{0.8}{1} + \frac{0.6}{2} + \frac{0.4}{3} + \frac{0.2}{4} \right\}$$

Fuzzy Intersection

- Commonly adopted method is using minimum
- Given two fuzzy sets *A* and *B* with membership functions $\mu_A(x)$ and $\mu_B(x)$, the intersection A and B defined over the same universe of discourse X is a new fuzzy set $A \cap B$ also on X with membership function which is the minimum of the grades of membership function of every x to A and B

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

EXAMPLE PROBLEM 3

- Find $A \cap B$

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

$$A \cap B = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5} \right\}$$

PROBLEM 4

- Find not Small and Not Very Large

Small $=\left\{\frac{0}{1}+\frac{0.36}{2}+\frac{0.64}{3}+\frac{0.84}{4}+\frac{0.96}{5}\right\}$

Very Large $=\left\{\frac{1}{1}+\frac{1}{2}+\frac{0.9}{3}+\frac{0.6}{4}\right\}$

$$\text{Solution}=\left\{\frac{0.1}{3}+\frac{0.16}{4}+\frac{0.04}{5}\right\}$$

Fuzzy Union

- Most common method for fuzzy union is to take maximum
- Given two fuzzy sets *A* and *B* with membership functions $\mu_A(x)$ and $\mu_B(x)$, the union A and B defined over the same universe of discourse X is a new fuzzy set A∪B also on X with membership function which is the maximum of the grades of membership function of every x to A and B
- $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

EXAMPLE PROBLEM 5

Let $X = \{ 1,2,3,4,5,6,7\}$

$A = \{ (3, 0.7), (5, 1), (6, 0.8) \}$ and

$B = \{(3, 0.9), (4, 1), (6, 0.6) \}$

Find $A \cap B$, $A \cup B$, A' and $B - A$

$A \cap B = \{ (3, 0.7), (6, 0.6) \}$

$A \cup B = \{ (3, 0.9), (4, 1), (5, 1), (6, 0.8) \}$

$A' = \{(1, 1),(2, 1), (3, 0.3), (4, 1), (6, 0.2),(7, 1)\}$

$B\text{-}A = B \cap A' = \{ (3, 0.3), (4, 1), (6, 0.2)\}$

EXAMPLE PROBLEM 6

- Find Small ∩ Large, Small ∪ Large

$$If \qquad U = \{0.01, 0.1, 0.5, 0.8, 1, 2, 3, 4, 5, 10\}$$

$$Small = \left\{ \frac{1}{0.01} + \frac{0.9}{0.1} + \frac{0.5}{0.5} + \frac{0.2}{0.8} + \frac{0.1}{1} + \frac{0.01}{2} \right\}$$

$$\text{Large} = \left\{ \frac{0.25}{1} + \frac{0.3}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.75}{5} + \frac{1}{10} \right\}$$

$$\text{Small} \cap \text{Large} = \left\{ \frac{0.1}{1} + \frac{0.01}{2} \right\}$$

$$\text{small} \cup \text{Large} = \left\{ \frac{1}{0.1} + \frac{0.9}{0.1} + \frac{0.5}{0.5} + \frac{0.2}{0.8} + \frac{0.25}{1} + \frac{0.3}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.75}{5} + \frac{1}{10} \right\}$$

Example Problem 7

Find A ∩ B, A ∪ B and A' given

$$A = \left\{ \frac{0.4}{1} + \frac{0.6}{2} + \frac{0.7}{3} + \frac{0.8}{4} \right\} \quad B = \left\{ \frac{0.3}{1} + \frac{0.65}{2} + \frac{0.4}{3} + \frac{0.1}{4} \right\}$$

$$A \cap B = \left\{ \frac{0.3}{1} + \frac{0.6}{2} + \frac{0.4}{3} + \frac{0.1}{4} \right\}$$

$$A \cup B = \left\{ \frac{0.4}{1} + \frac{0.65}{2} + \frac{0.7}{3} + \frac{0.8}{4} \right\}$$

$$\overline{A} = \left\{ \frac{0.6}{1} + \frac{0.4}{2} + \frac{0.3}{3} + \frac{0.2}{4} \right\}$$

EXAMPLE PROBLEM 8

Given two fuzzy sets A and B

- Calculate the of union of the set A and set B
- Calculate the intersection of the set A and set B
- Calculate the complement of the union of A and B

$$A = \left\{ \frac{0.0}{-2} + \frac{0.3}{-1} + \frac{0.6}{0} + \frac{1.0}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0.0}{4} \right\}$$

$$B = \left\{ \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.7}{0} + \frac{1.0}{1} + \frac{0.5}{2} + \frac{0.2}{3} + \frac{0.0}{4} \right\}$$

$$A \cup B = \left\{ \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.7}{0} + \frac{1.0}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0.0}{4} \right\}$$

$$A \cap B = \left\{ \frac{0.0}{-2} + \frac{0.3}{-1} + \frac{0.6}{0} + \frac{1.0}{1} + \frac{0.5}{2} + \frac{0.2}{3} + \frac{0.0}{4} \right\}$$

$$\overline{(A \cup B)} = \left\{ \frac{0.9}{-2} + \frac{0.6}{-1} + \frac{0.3}{0} + \frac{0.0}{1} + \frac{0.4}{2} + \frac{0.7}{3} + \frac{1.0}{4} \right\}$$

BOUNDED DIFFERENCE

- For two fuzzy sets A, B the bounded difference $A \odot B$ is given by

$$\mu_{AOB(x)} = \text{Max}\ [0, \mu_{A(x)} - \mu_{B(x)}] \text{ for all } x \in X$$

BOUNDED DIFFERENCE – EXAMPLE PROBLEM 9

- Find A ⊙ B $\mu_{A \odot B(x)} = \text{Max}\,[0, \mu_{A(x)} - \mu_{B(x)}]$

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

$$A \odot B = \left\{ \frac{0.5}{2} + \frac{0}{3} + \frac{0.1}{4} + \frac{0}{5} \right\}$$

BOUNDED SUM

- For two fuzzy sets A, B the bounded sum $A \oplus B$ is given by

$$\mu_{A \oplus B} = \min(1, \mu_A(x) + \mu_B(x)) \text{ for all } x \in X$$

BOUNDED SUM–EXAMPLE PROBLEM 11

- Find $A \oplus B$

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

$$A \oplus B = \left\{ \frac{1}{2} + \frac{1}{3} + \frac{0.5}{4} + \frac{0.6}{5} \right\}$$

Algebraic Product

- Product of two fuzzy sets A and B defined on the same universe of discourse X is a new fuzzy set A·B, with membership function that equals to the algebraic product of the membership function of A and B
- $\mu_{A.B}(x) \equiv \mu_{A}(x) \cdot \mu_{B}(x)$

PRODUCT – EXAMPLE PROBLEM 10

- Find A.B

$$A=\left\{\frac{1}{2}+\frac{0.5}{3}+\frac{0.3}{4}+\frac{0.2}{5}\right\}$$

$$B=\left\{\frac{0.5}{2}+\frac{0.7}{3}+\frac{0.2}{4}+\frac{0.4}{5}\right\}$$

$$A\bullet B=\left\{\frac{0.5}{2}+\frac{0.35}{3}+\frac{0.06}{4}+\frac{0.08}{5}\right\}$$

ALGEBRAIC SUM

- for all x€ X

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) * \mu_B(x)$$

ALGEBRAIC SUM–EXAMPLE PROBLEM 12

- Find A+B

$$A=\left\{\frac{1}{2}+\frac{0.5}{3}+\frac{0.3}{4}+\frac{0.2}{5}\right\}$$

$$B=\left\{\frac{0.5}{2}+\frac{0.7}{3}+\frac{0.2}{4}+\frac{0.4}{5}\right\}$$

$$A+B=\left\{\frac{1}{2}+\frac{0.85}{3}+\frac{0.44}{4}+\frac{0.52}{5}\right\}$$

EXAMPLE PROBLEM 13

- Find algebraic sum, algebraic product, bounded sum and bounded difference

$$A=\left\{\frac{0.2}{1}+\frac{0.3}{2}+\frac{0.4}{3}+\frac{0.5}{4}\right\} B=\left\{\frac{0.1}{1}+\frac{0.2}{2}+\frac{0.2}{3}+\frac{1}{4}\right\}$$

$$A+B=\left\{\frac{0.28}{1}+\frac{0.44}{2}+\frac{0.52}{3}+\frac{1}{4}\right\} \quad A.B=\left\{\frac{0.02}{1}+\frac{0.06}{2}+\frac{0.08}{3}+\frac{0.5}{4}\right\}$$

$$A\oplus B=\left\{\frac{0.3}{1}+\frac{0.5}{2}+\frac{0.6}{3}+\frac{1}{4}\right\} \quad A\bullet B=\left\{\frac{0.1}{1}+\frac{0.1}{2}+\frac{0.2}{3}+\frac{0.0}{4}\right\}$$

FUZZY SUBSET - RESULT

$$A \subseteq B \text{ iff } A \cap B = A \text{ and } A \cup B = B \text{ for any } \mathrm{A}, \mathrm{B} \in \mathrm{P}(\mathrm{X})$$

EMPTY FUZZY SET

- A fuzzy set A is called empty (denoted by $A = \emptyset$) if its membership function is zero everywhere in its universe of discourse X.
- $A \equiv \emptyset \qquad \text{if } \mu_A(x) = 0, \; \forall x \in X$

Equality of Fuzzy Sets

- Two fuzzy sets are said to be equal if their membership functions are equal for every element in the universe of discourse, that is
- $A \equiv B \quad \text{if } \mu_A(x) = \mu_B(x)\ \forall x \in X$

CARDINALITY

- Cardinality of a set is the total number of elements in that set

PROPERTIES OF FUZZY SETS

1. Commutativity

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}$$
$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$$

2. Associativity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C}$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}$$

3. Distributivity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C})$$

4. Idempotency

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A}$$
$$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

5. Identity

$$\underset{\sim}{A} \cup \phi = \underset{\sim}{A} \text{ and } \underset{\sim}{A} \cup U = U \text{(universal set)}$$
$$\underset{\sim}{A} \cap \phi = \phi \text{ and } \underset{\sim}{A} \cap U = \underset{\sim}{A}$$

PROPERTIES OF FUZZY SETS

6. Involution (double negation)

$$\bar{\bar{\underset{\sim}{A}}} = \underset{\sim}{A}$$

7. Transitivity

$$\text{If } \underset{\sim}{A} \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{C}, \text{ then } \underset{\sim}{A} \subseteq \underset{\sim}{C}$$

8. Demorgan's law

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$$

$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$$

Fuzzy Logic Laws-Verification

- **Obeys Demorgan's Laws**

$$\mu_{\overline{(A \cap B)}}(x) = \mu_{\overline{A} \cup \overline{B}}(x)$$

$$1 - \min(\mu_A(x), \mu_B(x)) = \max[(1 - \mu_A(x)), (1 - \mu_B(x))]$$

VERIFICATION WITH EXAMPLE

- Verify $\mu_{\overline{(A\cap B)}}(x) = \mu_{\bar{A}\cup\bar{B}}(x)$

$$A = \left\{\frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5}\right\}$$

$$B = \left\{\frac{0.5}{2} + \frac{1}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$$

$$\mu_{\overline{(A\cap B)}}(x) = \left\{\frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5}\right\}$$

Fuzzy Logic Laws – Verification Contd....

- Fails in Law of excluded middle

$\mathbf{A} \cup \overline{\mathbf{A}} = \mathbf{X}$ is not true

Consider $A = \left\{ \frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0.5}{4} \right\}$

$$\overline{A} = \left\{ \frac{0.8}{1} + \frac{0.7}{2} + \frac{0.6}{3} + \frac{0.5}{4} \right\}$$

$$A \cup \overline{A} = \left\{ \frac{0.8}{1} + \frac{0.7}{2} + \frac{0.6}{3} + \frac{0.5}{4} \right\}$$

Fuzzy Logic Laws- Verification Contd..

- Fails in Law Contradiction

$$A \cap \overline{A} \neq \phi$$

- Thus, (the set of numbers *close* to 2) AND (the set of numbers not *close* to 2) ≠ null set

THE LAW OF CONTRADICTION–VIOLATION - PROOF

- The law of contradiction $A \cap \overline{A} \neq \phi$
- To verify that the law of contradiction is violated for fuzzy sets, we need only to show that

  $\min[\mu_A(x), 1-\mu_A(x)]=0$ is not true for any $x \in X$
- This can be proved easily since the equation is obviously violated for all values of $\mu_A(x)$ other than 0 and 1and is satisfied only by $\mu_A(x) \in \{0,1\}$.
- if $\mu_A(x)=0.2$ $\min[\mu_A(x), 1-\mu_A(x)]=0.2$ #0 so proved

PROOF WITH EXAMPLE

$$A \cap \overline{A} \neq \phi$$

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$\overline{A} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} \right\}$$

$$A \cap \overline{A} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \neq \phi$$

LAW OF ABSORPTION-PROOF

- To verify the law of absorption, $A \cup (A \cap B) = A$
  - Requires $\max[\mu_A(x), \min(\mu_A(x), \mu_B(x))] = \mu_A(x)$ is satisfied for all $x \in X$
  - Consider two cases:
  - (1) $\mu_A(x) \leq \mu_B(x)$

$\Rightarrow \max[\mu_A(x), \min(\mu_A(x), \mu_B(x))] = \max(\mu_A(x), \mu_A(x)) = \mu_A(x)$

  - (2) $\mu_A(x) > \mu_B(x)$

$\Rightarrow \max[\mu_A(x), \min(\mu_A(x), \mu_B(x))] = \max[\mu_A(x), \mu_B(x)] = \mu_A(x)$

  - Both cases verified

OTHER RESULTS

- $A \cup \bar{A} \neq \mathrm{X}$
- $A \cap \emptyset = \emptyset$
- $A \cup \emptyset = A$
- $A \cap X = A$
- $A \cup X = \mathrm{X}$

FUZZY RELATION

CLASSICAL RELATION AND FUZZY RELATION

- Mapping between 2 sets
- Presence or absence of a connection or association between elements of 2 sets

CRISP RELATION

- A *relation* among crisp sets A_1, A_2, A_3, … A_n is a subset of the Cartesian product. It is denoted by

$$R \subseteq A_1 \times A_2 \times \ \ldots \times A_n$$

Cartesian Product

- Let A and B are two non-empty sets, then the Cartesian Product AxB is
- $A \times B = \{(a,b) / a \in A, b \in B\}$
- $A=\{a1,a2\}$ $B=\{b1,b2\}$ $C=\{c1,c2\}$
- $A \times B = \{(a1,b1),(a1,b2),(a2,b1),(a2,b2)\}$
- $B \times C = \{(b1,c1),(b1,c2),(b2,c1),(b2,c2)\}$

CARTESIAN PRODUCT OF A, B,C

- If $A = \{a1, a2\}$ $B = \{b1, b2\}$

$C = \{c1, c2\}$ find $A \times B \times C$

A x B x C = {(a1,b1,c1), (a1,b1,c2), (a1,b2,c1), (a1,b2,c2), (a2,b1,c1), (a2,b1,c2), (a2,b2,c1), (a2,b2,c2)}

ASSOCIATING MEMBERSHIP

- Using the membership function defines the crisp relation R :

$$\mu_R(x_1,\ x_2,\ \ldots,x_n) = \begin{cases} 1 & \text{iff } (x_1,\ x_2,\ \ldots,\ x_n) \in R, \\ 0 & \text{otherwise} \end{cases}$$

where $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$

Fuzzy Relation

- A *fuzzy relation* on the Cartesian product of crisp sets $A_1, A_2, ..., A_n$ where tuples $(x_1, x_2, ..., x_n)$ have varying memberships within the relation
- The membership grade indicates the strength of the relation present between the elements of the tuple

$$\mu_R : A_1 \times A_2 \times ... \times A_n \to [0,1]$$

$$R = \left\{((x_1, x_2, ..., x_n), \mu_R) \mid \mu_R(x_1, x_2, ..., x_n) \geq 0, x_1 \in A_1, x_2 \in A_2, ..., x_n \in A_n\right\}$$

MATRIX REPRESENTATION

- .

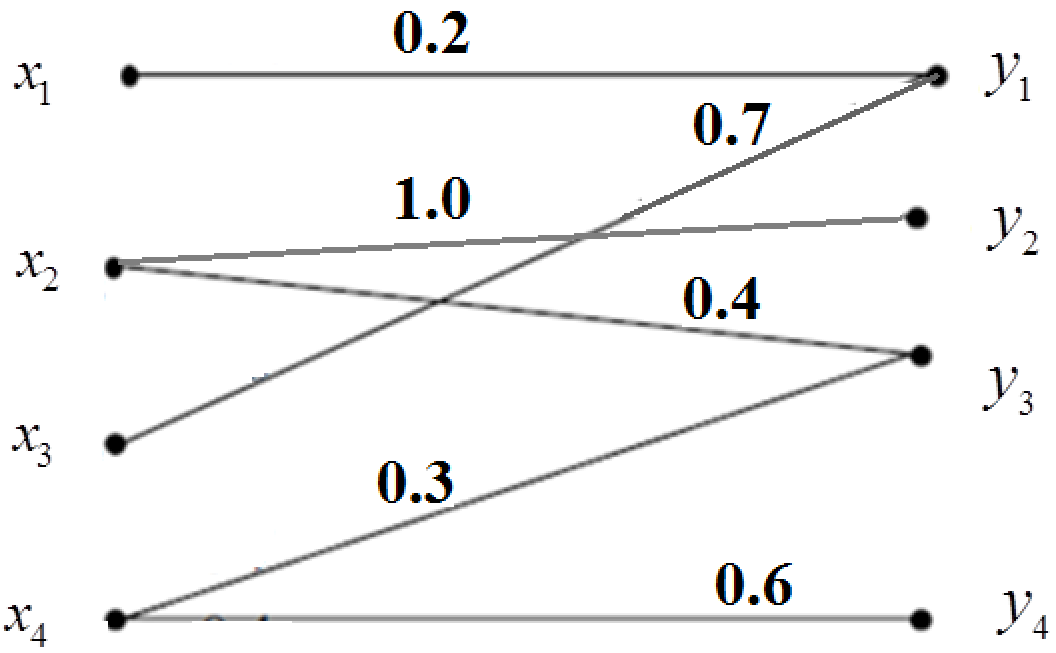
A \ B	y_1	y_2	y_3	y_4
x_1	1.0	0.0	0.0	0.0
x_2	0.0	1.0	1.0	0.0
x_3	1.0	0.0	0.0	0.0
x_4	0.0	0.0	1.0	1.0

Crisp

A \ B	y_1	y_2	y_3	y_4
x_1	0.2	0.0	0.0	0.0
x_2	0.0	1.0	0.4	0.0
x_3	0.7	0.0	0.0	0.0
x_4	0.0	0.0	0.3	0.6

Fuzzy

FUZZY RELATION



Fuzzy Relation

Operations on Relations

- Union
- Intersection
- Complement
- Composition

UNION RELATION

- Union

$$\mu_{R\cup S}(x,y) = \max(\mu_R(x,y), \mu_s(x,y)) \qquad \forall (x,y) \in A \times B$$

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R\cup S}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

INTERSECTION RELATION

$$\mu_{R\cap S}(x,y) = \min(\mu_R(x,y), \mu_s(x,y)) \qquad \forall (x,y) \in A \times B$$

- Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R\cap S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

OPERATIONS ON FUZZY RELATIONS

- Complement relation:

$$\forall (x, y) \in A \times B$$

$$\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$$

- Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

$M_{\bar{R}}$	a	b	c
1	0.7	0.8	0.0
2	0.2	0.0	0.0
3	1.0	0.0	1.0

Composition of fuzzy relations

- Let R and S are relations defined AxB and BxC respectively

$R \cdot S = \{(a,c)/(a,c) \in AxC$ such that $b \in B$ and

$(a,b) \in R$ and $(b,c) \in S\}$

COMPOSITION OF FUZZY RELATIONS

- Max-min composition

$$\forall (x, y) \in A \times B,\ \forall (y, z) \in B \times C$$

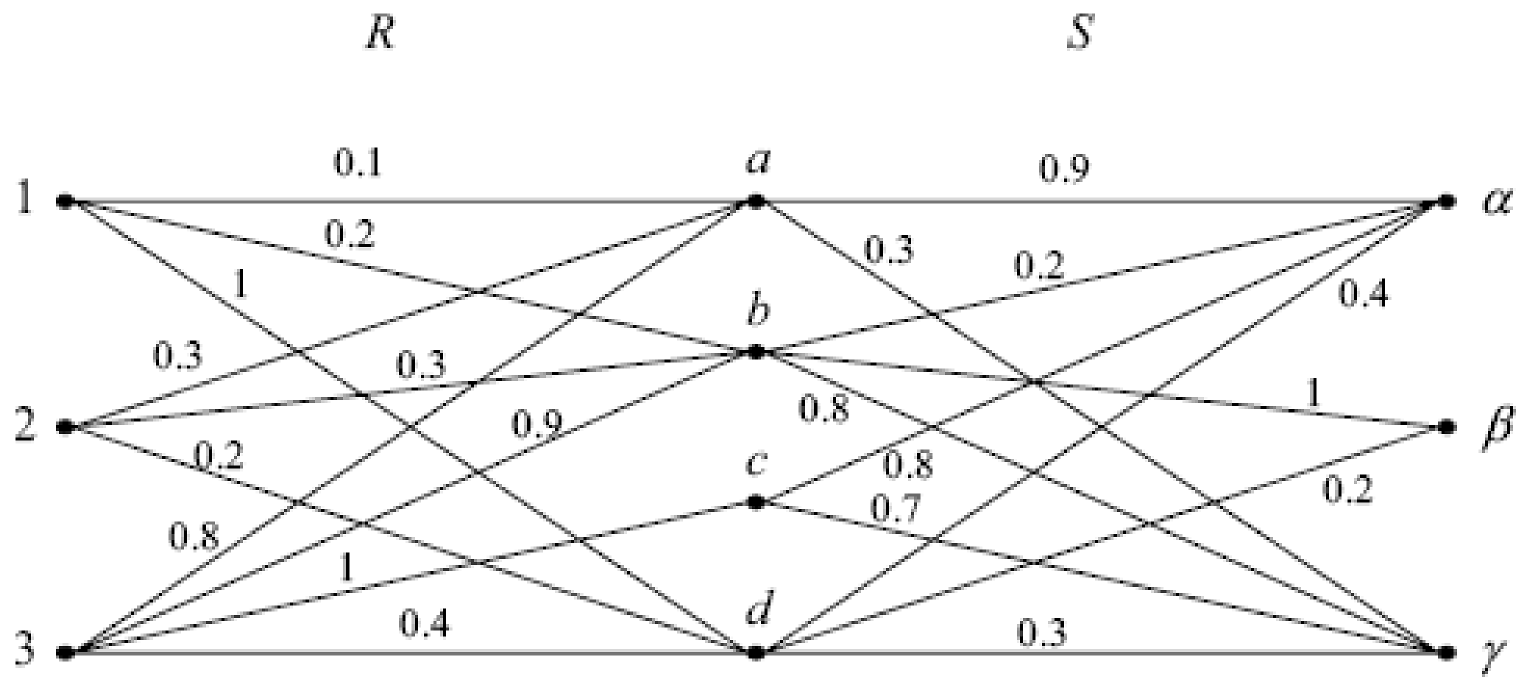
$$\mu_{R \bullet S}(x, z) = \max_{y}[\min(\mu_R(x, y), \mu_S(y, z))]$$

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

COMPOSITION OF FUZZY RELATIONS



COMPOSITION OF FUZZY RELATIONS

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{R.S}(1,\ \alpha) = \max[\min(0.1, 0.9), \min(0.2, 0.2), \min(0.0, 0.8), \min(1.0, 0.4)]$$

$$= \max[0.1, 0.2, 0.0, 0.4] = 0.4$$

Composition of fuzzy relations

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

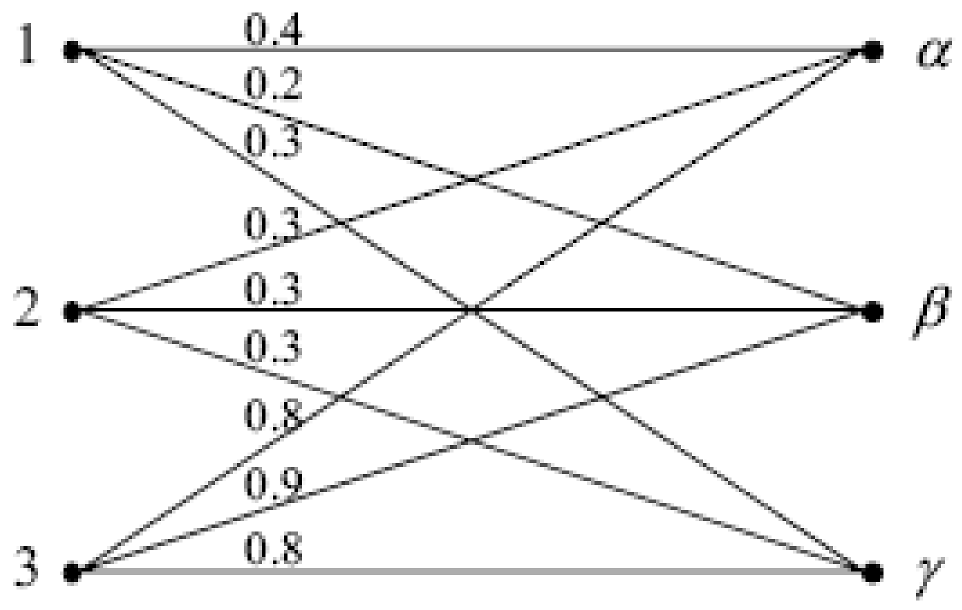
$$\mu_{R.S}(1,\ \beta) = \max[\min(0.1, 0.0), \min(0.2, 1.0), \min(0.0, 0.0), \min(1.0, 0.2)]$$

$$= \max[0.0, 0.2, 0.0, 0.2] = 0.2$$

COMPOSITION OF FUZZY RELATIONS

R • S	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

R • S



MAX STAR COMPOSITION

- Max product: C = A*B=$\max(a_{ik} * b_{kj})$ where * stands for any binary operator
- Example

$$R = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.7 & 0.5 \\ x_2 & 0.8 & 0.4 \end{array} \qquad S = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0.9 & 0.6 & 0.2 \\ y_2 & 0.1 & 0.7 & 0.5 \end{array}$$

COMPUTING MAX PRODUCT

$$R = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.7 & 0.5 \\ x_2 & 0.8 & 0.4 \end{array}$$

$$S = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0.9 & 0.6 & 0.2 \\ y_2 & 0.1 & 0.7 & 0.5 \end{array}$$

$T(x1,z1)=\max[\ R(x_1\ ,y_1)*S\ (y_1,z_1),\ R(x_1\ ,y_2)*S\ (y_2,z_1)$

$= \max(0.7*0.9,0.5*0.1)=\max(0.63,0.05)=0.63$

Computing Max Product

$$R = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.7 & 0.5 \\ x_2 & \boxed{0.8} & \boxed{0.4} \end{array} \qquad S = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0.9 & \boxed{0.6} & 0.2 \\ y_2 & 0.1 & \boxed{0.7} & 0.5 \end{array}$$

T(x2,z2)=max[$R(x_2, y_1)*S(y_1,z_2)$, $R(x_2, y_2)*S(y_2,z_2)$

= max(0.8*0.6,0.4*0.7)=max(0.48,0.28)=0.48

MAX PRODUCT- RESULT

$$R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$T = \begin{array}{c} x_1 \\ x_2 \end{array} \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix}$$

OPERATIONS ON FUZZY RELATION

1. Union:

$$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]$$

2. Intersection:

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]$$

3. Complement:

$$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$$

4. Containment:

$$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$$

FUZZY CARTESIAN PRODUCT

- Cartesian product of two fuzzy sets A and B, AxB
- Membership function

The membership function of fuzzy relation is given by

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x, y) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

EXAMPLE 1

- If $$A = \left\{ \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \right\}$$
- $$B = \left\{ \frac{0.3}{y_1} + \frac{0.9}{y_2} \right\}$$

Find A × B

$$A \times B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} Y_1 & \quad Y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

EXAMPLE 2

- If Good- Service is given by

$$GS = \left\{ \frac{1}{a} + \frac{0.8}{b} + \frac{0.6}{c} + \frac{0.4}{d} + \frac{0.2}{e} \right\}$$

- Where a,b,c,d,e are service ratings and Satisfied is

$$S = \left\{ \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} \right\}$$

- Where satisfaction levels are 1,2,3,4,5 Find GS × S

$$GS \times S = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\ 0.2 & 0.4 & 0.6 & 0.8 & 0.8 \\ 0.2 & 0.4 & 0.6 & 0.6 & 0.6 \\ 0.2 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

EXAMPLE PROBLEM

- For speed control of DC motor, the membership functions for series resistance, armature current and speed are given as

$$R=\left\{\frac{0.4}{30}+\frac{0.6}{60}+\frac{1.0}{100}+\frac{0.1}{120}\right\}$$

$$I=\left\{\frac{0.2}{20}+\frac{0.3}{40}+\frac{0.6}{60}+\frac{0.8}{80}+\frac{1.0}{100}+\frac{0.2}{120}\right\}$$

$$N=\left\{\frac{0.35}{500}+\frac{0.67}{1000}+\frac{0.97}{1500}+\frac{0.25}{1800}\right\}$$

Compute relation T relating resistance to motor speed

SOLUTION

- T=RXN

$$T = \begin{bmatrix} 0.35 & 0.4 & 0.4 & 0.25 \\ 0.35 & 0.6 & 0.6 & 0.25 \\ 0.35 & 0.67 & 0.97 & 0.25 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

EXAMPLE PROBLEM

- In soil mechanics, a fuzzy set Yis defined on three levels of compaction low, medium and high. Another fuzzy set Xis defined on a universe of soil gradations poor, moderate and uniform. Find R=AXB . Find C.R using max-min composition

$$A = \text{PoorGradiatedSoil} = \left\{ \frac{0.9}{x1} + \frac{0.4}{x2} + \frac{0.0}{x3} \right\} \quad B = \text{WellCompactedSoil} = \left\{ \frac{0.1}{y1} + \frac{0.7}{y2} + \frac{1}{y3} \right\}$$

$$C = \left\{ \frac{0.3}{x1} + \frac{1.0}{x2} + \frac{0.2}{x3} \right\}$$

SOLUTION

- .

$$R = A \times B = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.1 & 0.7 & 0.9 \\ x_2 & 0.1 & 0.4 & 0.4 \\ x_3 & 0.0 & 0.0 & 0.0 \end{array}$$

$$C.R = \begin{bmatrix} 0.3 & 1.0 & 0.2 \end{bmatrix} \bullet \begin{bmatrix} 0.1 & 0.7 & 0.9 \\ 0.1 & 0.4 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.4 & 0.4 \end{bmatrix}$$

MODULE 3

MEMBERSHIP FUNCTIONS

- Membership functions defines the fuzziness in a fuzzy set irrespective of the elements in the set, which are discrete or continuous
- Represented in graphical form

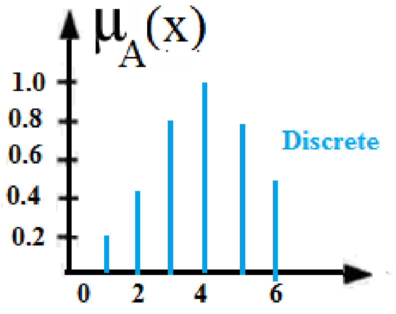
MEMBERSHIP FUNCTIONS

- Let us consider fuzzy set A, A = {(x, μA(x))| x Є X} where μA(x) is called the membership function for the fuzzy set A.
- X is referred to as the universe of discourse.
- The membership function associates each element x Є X with a value in the interval [0, 1].

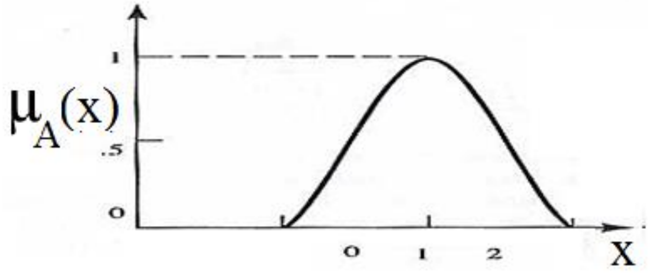
- The fuzzy set A can be alternatively denoted as follows:
- If X is discrete then A = Σ μA(xi) / xi
- If X is continuous then A = ∫ μA(x) / x

MEMBERSHIP FUNCTION

- The membership need not be described by discrete values



- Membership can be described by a continuous mathematical function



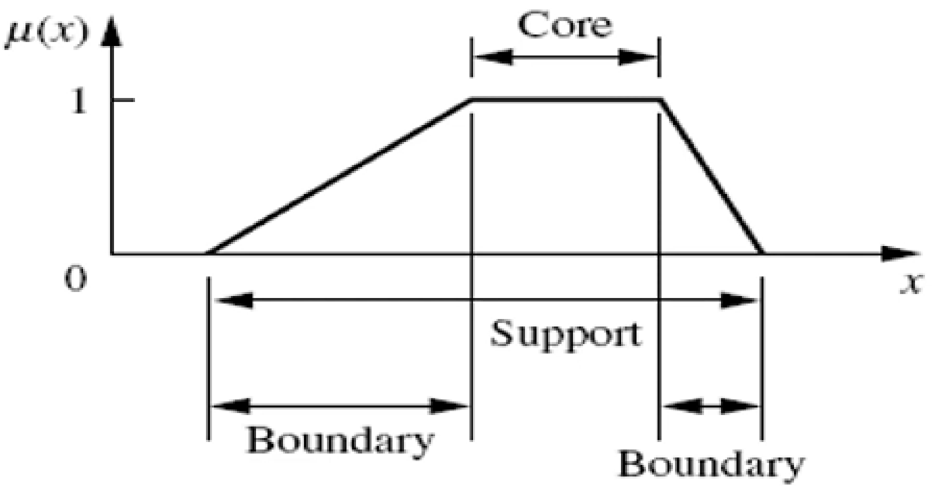
FEATURES OF MEMBERSHIP FUNCTION

- **1. Core :**
- The core of a membership function for some fuzzy set Ais defined as that region of the universe that is characterized by complete and full membership in the set A.
- That is, the core comprises those elements x of the universe such that $\mu A(x) = 1$.

- **2. Support :**
- The support of a membership function for some fuzzy set A is defined as that region of the universe that is characterized by nonzero membership in the set A.
- That is, the support comprises those elements x of the universe such that $\mu A(x) > 0$.

FEATURES OF MEMBERSHIP FUNCTION

- **3. Boundary:**
- The boundaries of a membership function for some fuzzy set Aa re defined as that region of the universe containing elements that have a nonzero membership but not complete membership.



FEATURES OF MEMBERSHIP FUNCTIONS

- **CORE:**

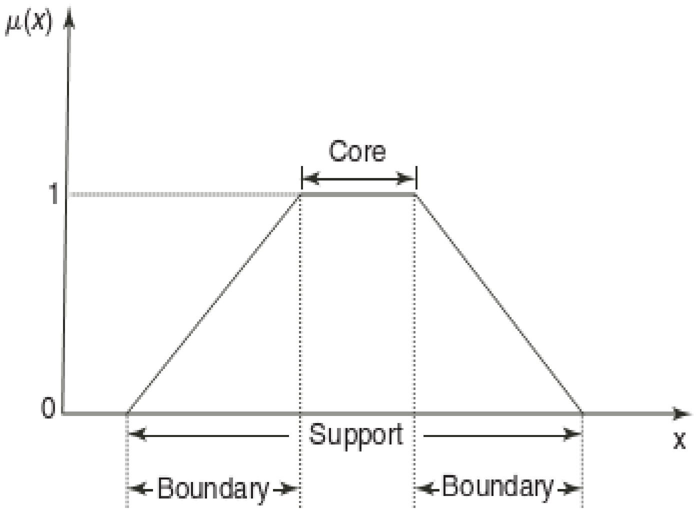
$$\mu_{\underset{\sim}{A}}(x) = 1$$

- **SUPPORT:**

$$\mu_{\underset{\sim}{A}}(x) > 0$$

- **BOUNDARY:**

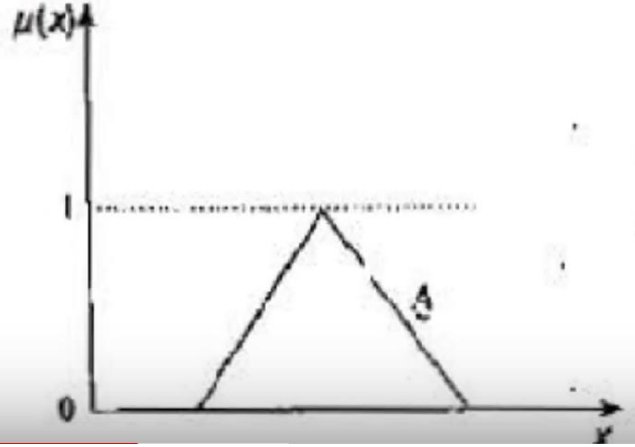
$$0 < \mu_{\underset{\sim}{A}}(x) < 1$$



NORMAL FUZZY SET

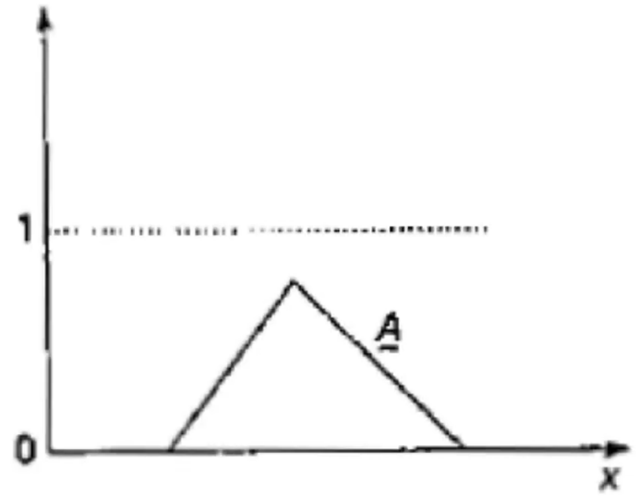
1) Normal Fuzzy set

A fuzzy set whose membership function has at least one elementx in the universe whose membership value is unity is called**normal fuzzy set**. The element for which the membership is equal to 1 is called prototypicalelement.



SUBNORMAL FUZZY

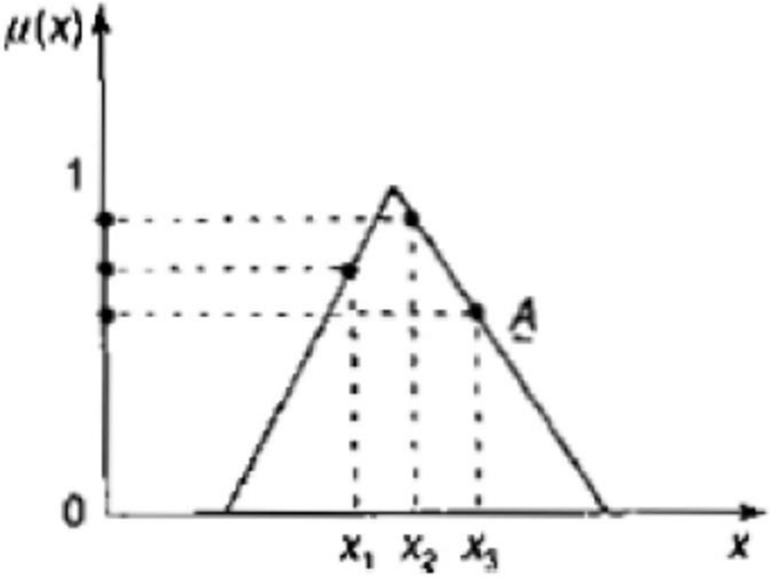
- **2) Subnormal Fuzzy set**
- A fuzzy set where in no membership function has its value equal to 1 is called **subnormal fuzzy set**.



CONVEX FUZZY SET

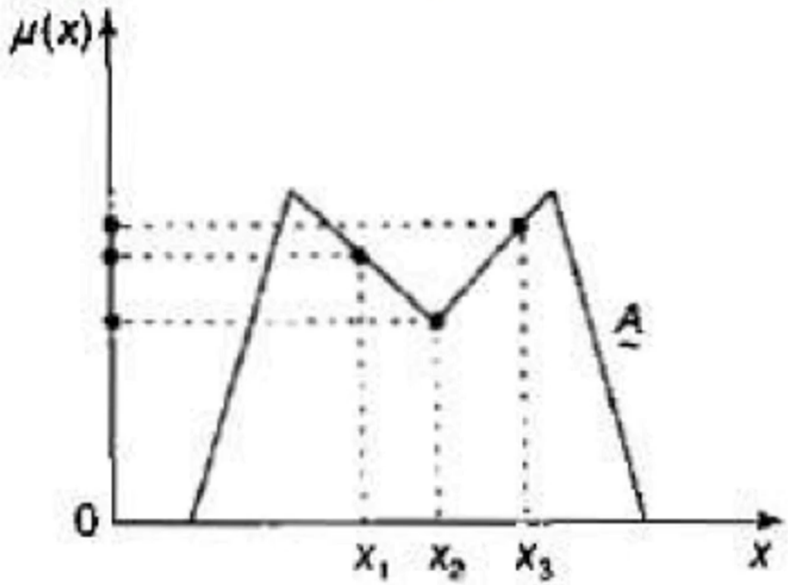
3) Convex Fuzzy set

A *convex fuzzy set* has a membership function whose membership values are strictly monotonically increasingor strictly monotonically decreasing or strictly monotonically increasing than strictly monotonically decreasingwithincreasing values for elements in the universe.



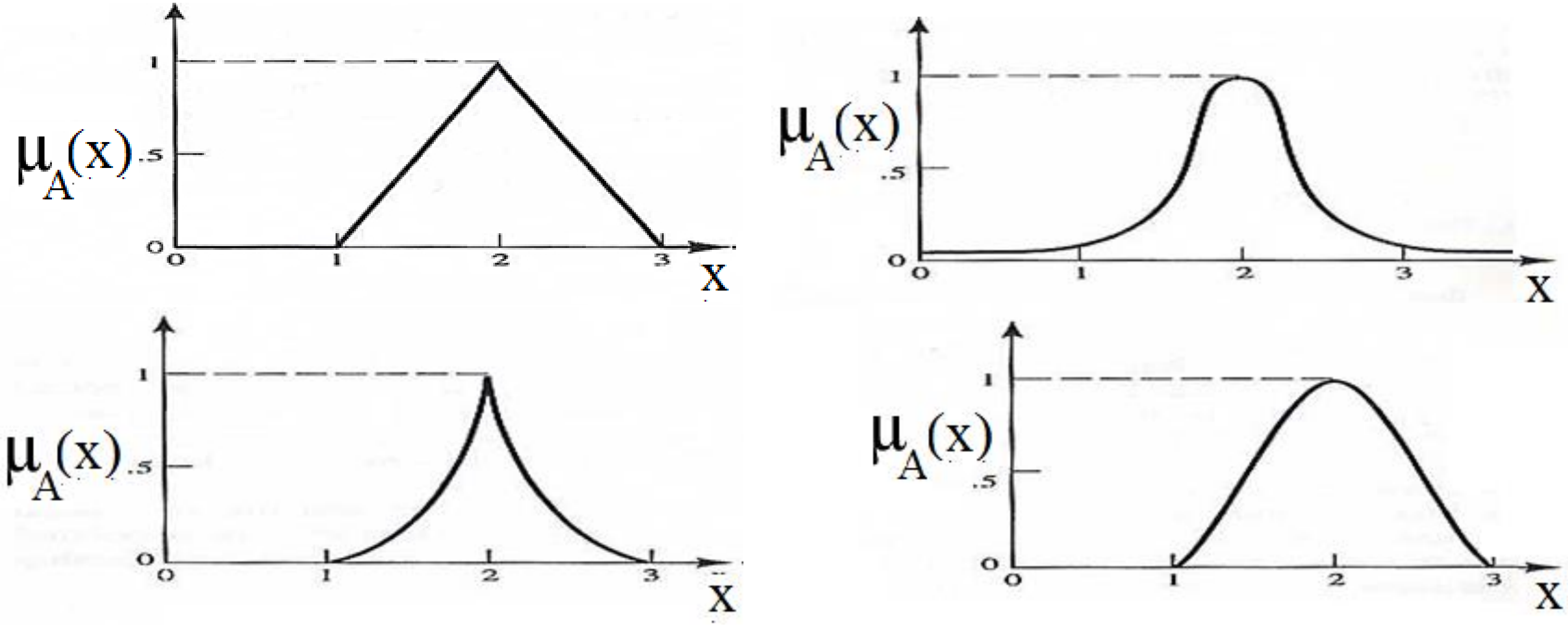
4) Nonconvex Fuzzy set

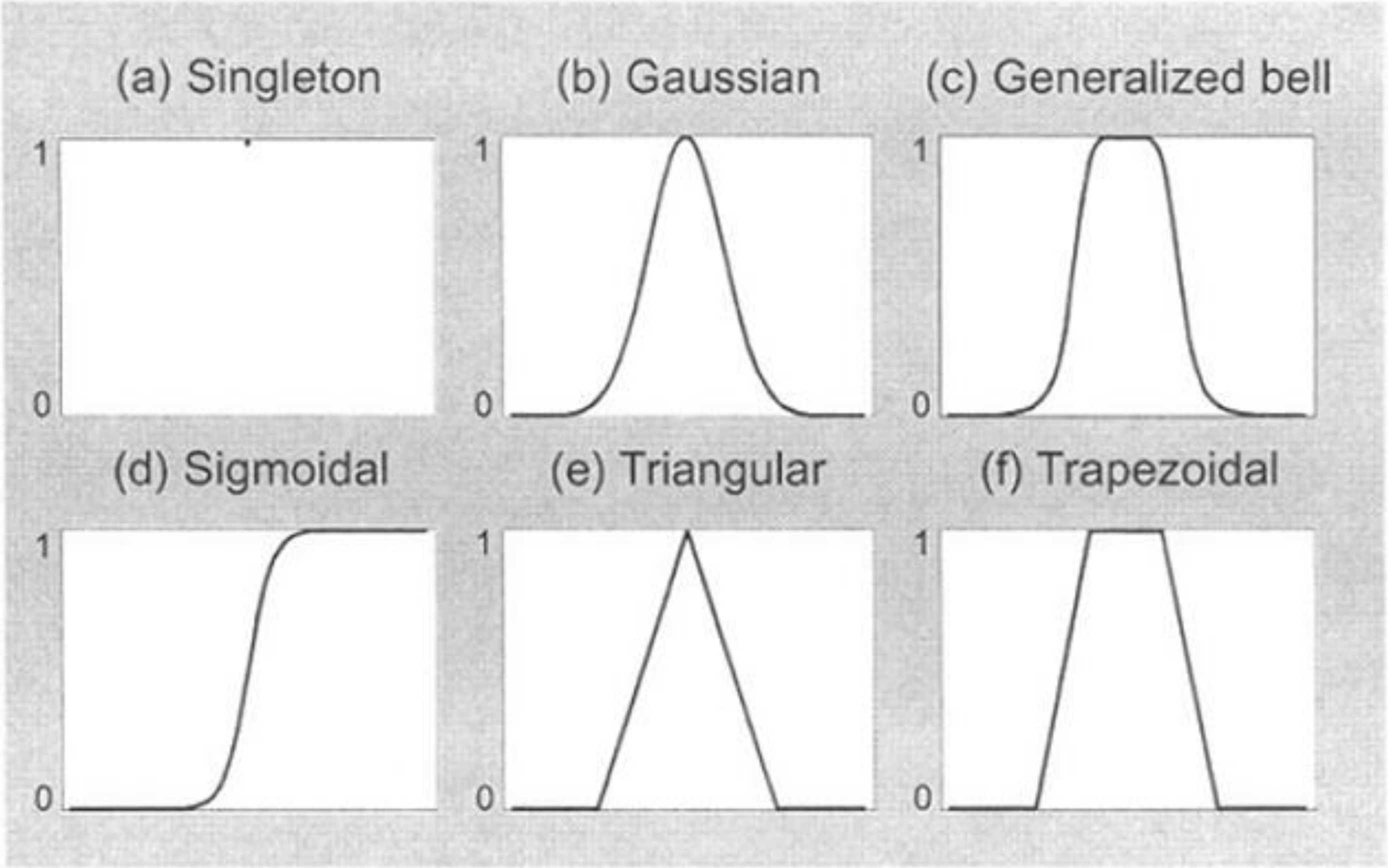
Afuzzy set possessing characteristics opposite to that of convex fuzzy set is called **nonconvexfuzzyset**, i.e., the membership values of the membership functionare not strictly monotonically increasing or decreasing or strictly monotonically increasing than decreasing.

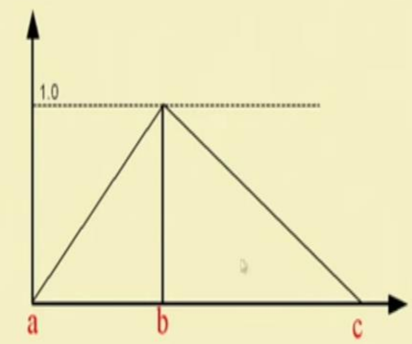


GRAPHICAL REPRESENTATION

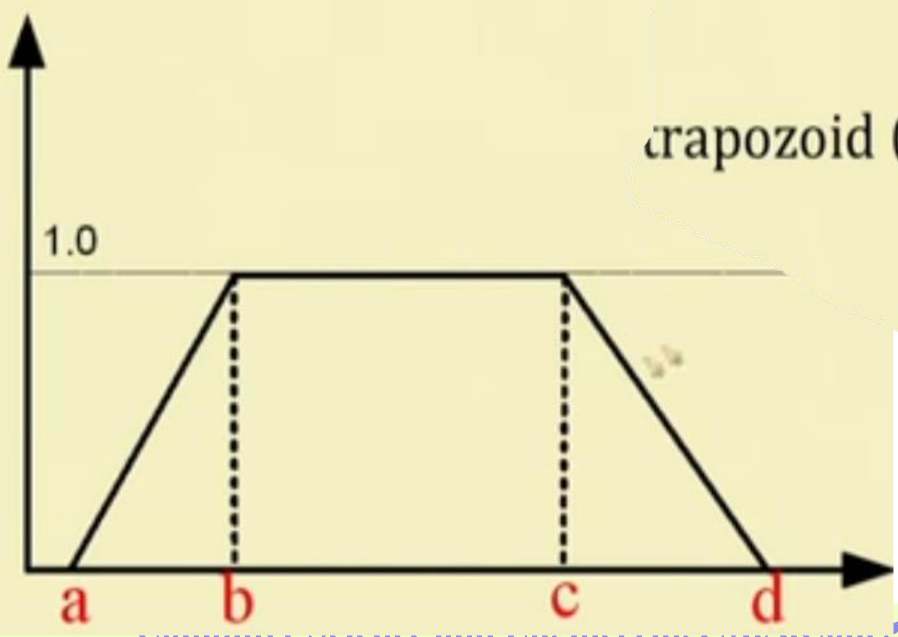
- Different shapes can be used for membership function such as triangular, trapezoidal and curved



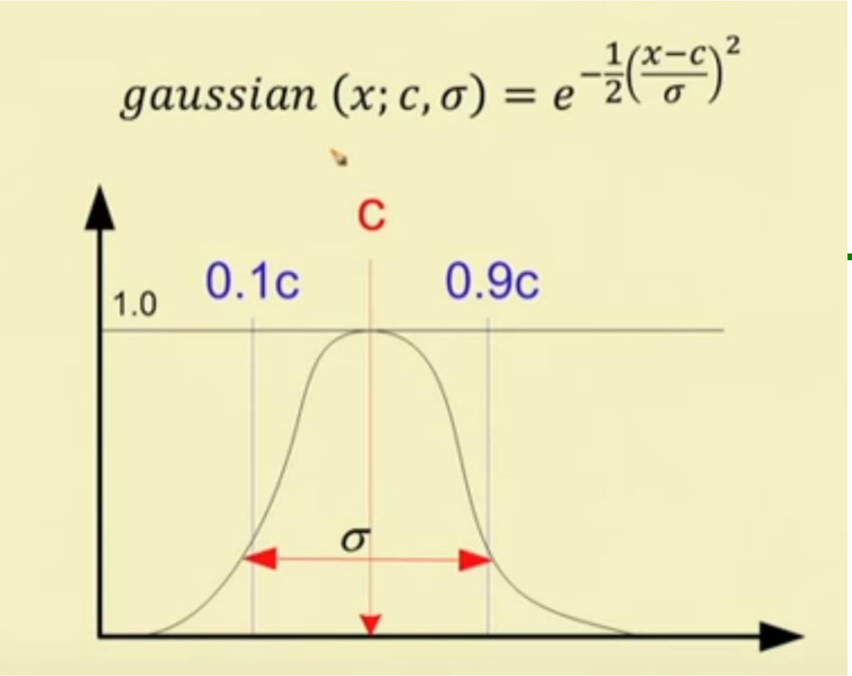

(a) Singleton
(b) Gaussian
(c) Generalized bell
(d) Sigmoidal
(e) Triangular
(f) Trapezoidal
1
0

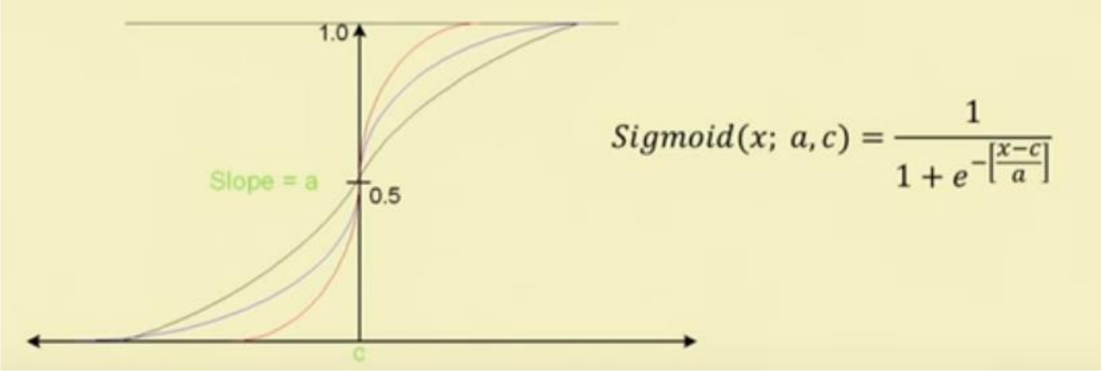


$$\text{triangle}(x;a,b,c)=\begin{Bmatrix} 0 & if\ x\le a \\ \dfrac{x-a}{b-a} & if\ a\le x\le b \\ \dfrac{c-x}{c-b} & if\ b\le x\le c \\ 0 & if\ c\le x \end{Bmatrix}$$



$$\text{trapozoid}(x;a,b,c,d)=\begin{Bmatrix} 0 & if\ x\le a \\ \dfrac{x-a}{b-a} & if\ a\le x\le b \\ 1 & if\ b\le x\le c \\ \dfrac{d-x}{d-c} & if\ c\le x\le d \\ 0 & if\ d\le x \end{Bmatrix}$$


$gaussian\,(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$
c
0.1c
0.9c
1.0
σ


1.0
Slope = a
0.5
c
$Sigmoid(x;\ a, c) = \frac{1}{1 + e^{-\left[\frac{x-c}{a}\right]}}$

FUZZY MEMBERSHIP FUNCTIONS

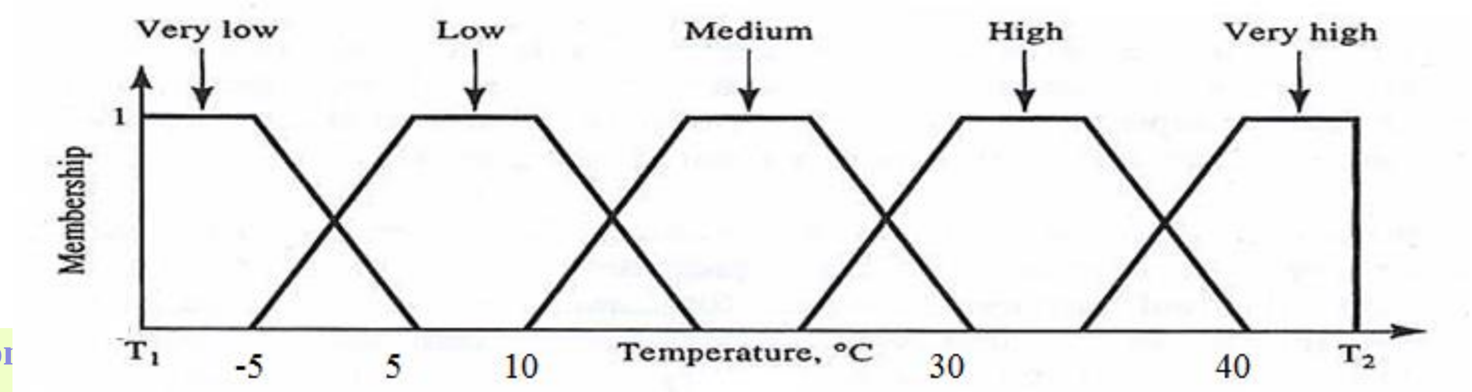
- Determining the membership function is a key issue in fuzzy set design
- Can be devised using previous experience
- Using machine learning techniques
  - Artificial neural networks, genetic algorithms, etc.
- Different shaped membership functions exist

Membership Value assignments - Methods

- Intuition
- Inference
- Rank Ordering
- Angular Fuzzy Sets
- Neural Networks
- Genetic algorithms
- Inductive Reasoning

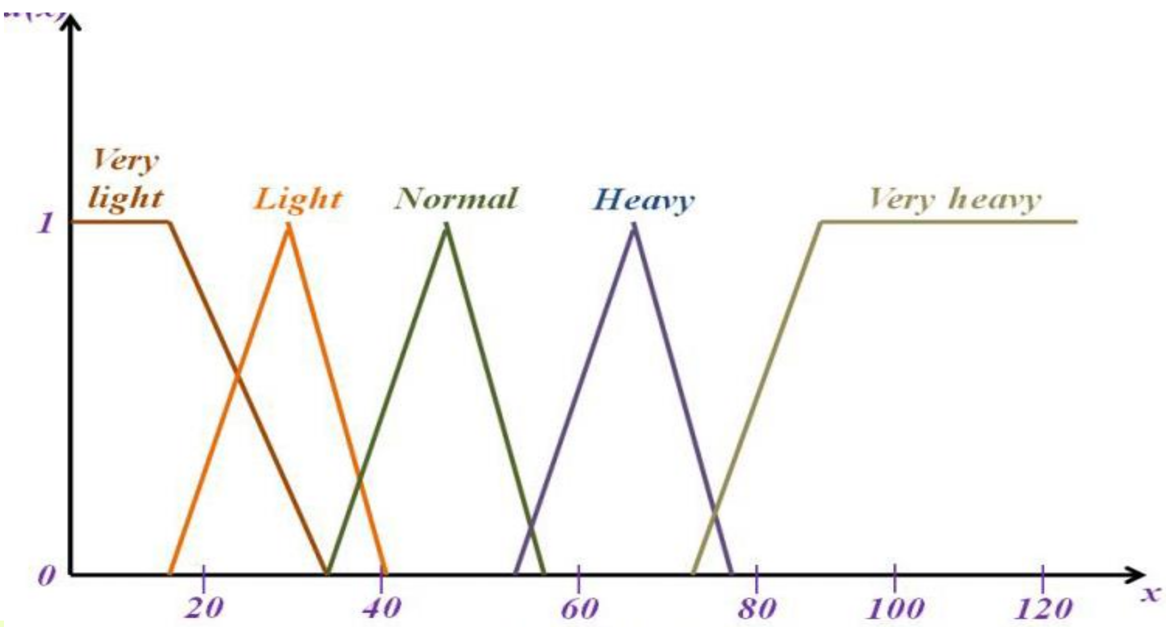
Intuition

- Application of innate intelligence and understanding of humans for finding(instinctive feeling)
- Involves contextual and semantic knowledge about an issue-Understanding about linguistic truth values
- Example fuzzy variable temperature
- Various shapes of universe of temperature



ANOTHER EXAMPLE-WEIGHT

- Using our own intuitions and definitions of universe of discourse, membership functions can be devised

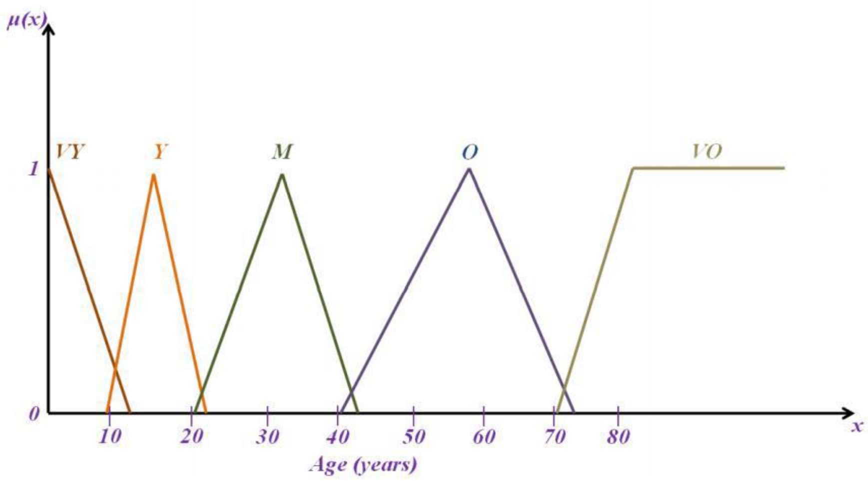


Weight in Kilogram

$$Very\ thin(VT) : W \leq 25$$
$$Thin(T) : 25 < W \leq 45$$
$$Average(AV) : 45 < W \leq 60$$
$$Stout(S) : 60 < W \leq 75$$
$$Very\ stout(VS) : W > 75$$

EXAMPLE 3-AGE

- Using our own intuitions and definitions of universe of discourse, membership functions can be devised

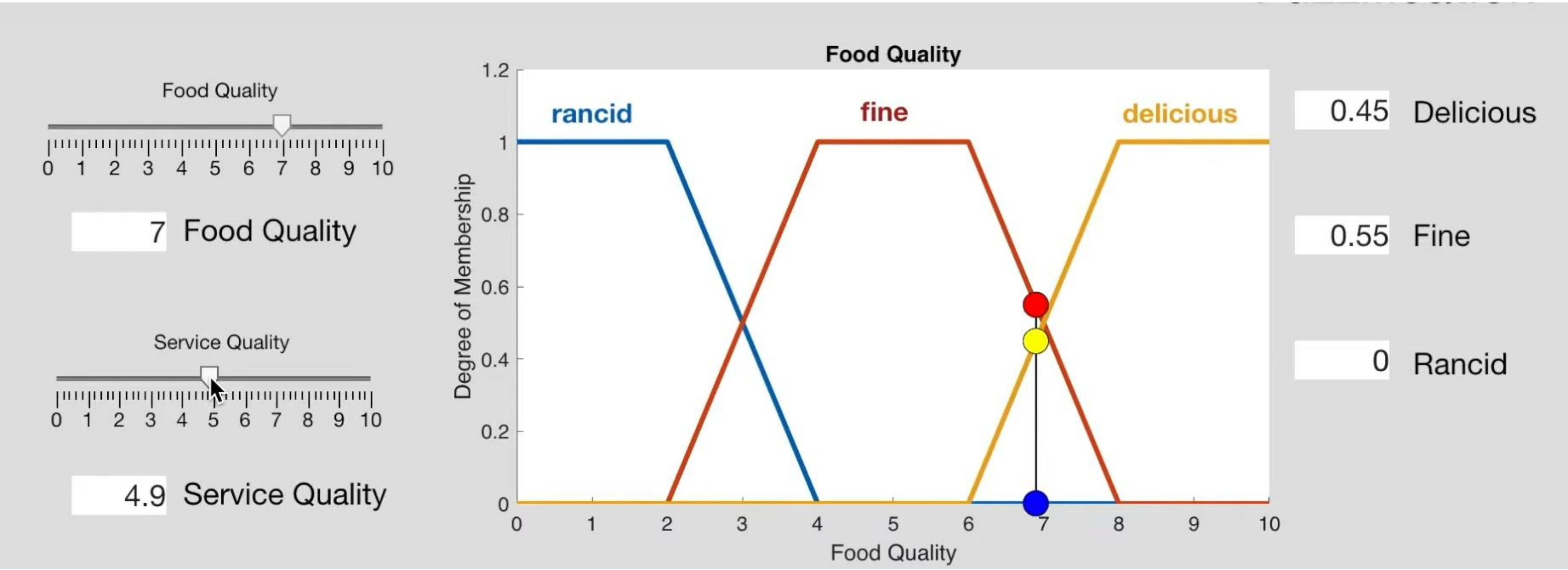


Linguistic Variables for Age

$$Very\ young(VY) : A < 12$$
$$Young(Y) : 10 \leq A \leq 22$$
$$Middle\ age(M) : 20 \leq A \leq 42$$
$$Old(O) : 40 \leq A \leq 72$$
$$Very\ old(VO) : 70 < A$$


Food Quality
0 1 2 3 4 5 6 7 8 9 10
1.2 Food Quality
Service Quality
0 1 2 3 4 5 6 7 8 9 10
0.1 Service Quality


Food Quality
0 1 2 3 4 5 6 7 8 9 10
7 Food Quality
Service Quality
0 1 2 3 4 5 6 7 8 9 10
4.9 Service Quality


Food Quality
0 1 2 3 4 5 6 7 8 9 10
7 Food Quality
Service Quality
0 1 2 3 4 5 6 7 8 9 10
4.9 Service Quality
Food Quality
rancid
fine
delicious
Degree of Membership
1.2
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10
Food Quality
0.45 Delicious
0.55 Fine
0 Rancid


Fuzzification
Food Quality
0 1 2 3 4 5 6 7 8 9 10
7 Food Quality
Service Quality
0 1 2 3 4 5 6 7 8 9 10
4.9 Service Quality
Food Quality
rancid
fine
delicious
Degree of Membership
0 0.2 0.4 0.6 0.8 1 1.2
0 1 2 3 4 5 6 7 8 9 10
Food Quality
0.45 Delicious
0.55 Fine
0 Rancid
Service Quality
poor
good
excellent
Degree of Membership
0 0.2 0.4 0.6 0.8 1 1.2
0 1 2 3 4 5 6 7 8 9 10
Service Quality
0 Excellent
0.99 Good
0.01 Poor

INFERENCE

- Uses deductive reasoning- infer a conclusion from available facts and knowledge
- Example to develop membership functions for isosceles triangle, right triangle, equilateral triangle
- Knowledge about geometry

Approximate Isosceles Triangle

- For an Isosceles Triangle -two angles are equal
- Assume $A \geq B \geq C \geq 0$

We know that $A + B + C = 180$

it can be inferred that If A=B or B=C, $\mu = 1$

ie if min(A-B,B-C)= 0 , $\mu = 1$

Approximate Isosceles Triangle

- Using the knowledge $A \geq B \geq C \geq 0$

and $A + B + C = 180$ membership function can be inferred as

$$\mu_I(A, B, C) = 1 - \frac{1}{60^\circ} \times \min(A - B, B - C)$$

If A=B or B=C $\mu_I = 1$

If A=120, B=60, C=0 $\mu_I = 0$

Approximate Right Triangle

- Using the knowledge $A \geq B \geq C \geq 0$

and $A + B + C = 180$ it can be inferred that $$\mu_R(A, B, C) = 1 - \frac{1}{90^\circ} \times |A - 90|$$

If A=90 $\mu_R = 1$

If A=180 $\mu_R = 0$

Isosceles Right Triangle

- Using the knowledge $A \geq B \geq C \geq 0$

and $A + B + C = 180$, it is inferred that

$$\mu_I(A,B,C) = 1 - \frac{1}{60^\circ} \times \min(A-B, B-C) \qquad \mu_R(A,B,C) = 1 - \frac{1}{90^\circ} \times |A - 90|$$

$$IR = I \cap R$$

$$\mu_{IR}(A,B,C) = \min[\mu_I(A,B,C), \mu_R(A,B,C)]$$

$$\mu_{IR}(A,B,C) = 1 - \max\left[\frac{1}{60^\circ} \times \min(A-B, B-C), \frac{1}{90^\circ} \times |A-90|\right]$$

$\mu_R = 1$ $\mu_R = 0$

Approximate Equilateral Triangle

- Using the knowledge $A \geq B \geq C \geq 0$ and $A + B + C = 180$ it can be inferred that

$$\mu_E(A, B, C) = 1 - \frac{1}{180^\circ} \times |A - C|$$

If A= B=C $\mu_E = 1$

If A=180,B=0, C=0 $\mu_E = 0$

Other Triangles

$$T = \overline{(I \cup R \cup E)} = \bar{I} \cap \bar{R} \cap \bar{E}$$

$$\mu_T(A,B,C) = \min\left(1-\mu_I(A,B,C), 1-\mu_E(A,B,C), 1-\mu_R(A,B,C)\right)$$

$$\mu(A,B,C) = \frac{1}{180^\circ} \times \min\left(3(A-B), 3(B-C), 2|A-90|, A-C\right)$$

EXAMPLE

- Find the membership of the following triangle with angles (A=85, B=50,C=45) in different types of triangles I,E,R and T

$$\mu_R = 0.94 \qquad \mu_I = 0.916$$

$$\mu_E = 0.7 \qquad \mu_T = 0.05$$

Rank Ordering Example

- Preferences by individuals, committee, poll and other methods
- Example membership for best color
- A questionnaire for pairwise preferences among 5 colors (red, orange, yellow, green, blue) – 1000 responses
- Total 1000*10 entries can be totaled and ranked

SAMPLE SURVEY

- Rank Ordering

	Red	Orange	Yellow	Green	Blue	Total	%	Rank
Red		517	525	545	661	2248	22.5	2
orange	483		841	477	576	2377	23.8	1
Yellow	475	159		534	614	1782	17.8	4
green	455	523	466		643	2087	20.9	3
Blue	339	424	386	357		1506	15	5
total						10000		

Membership best color

Blue=0.6,yellow 0.75 ,green 0.87,red=0.94,Orange=0.99

ANGULAR FUZZY SETS

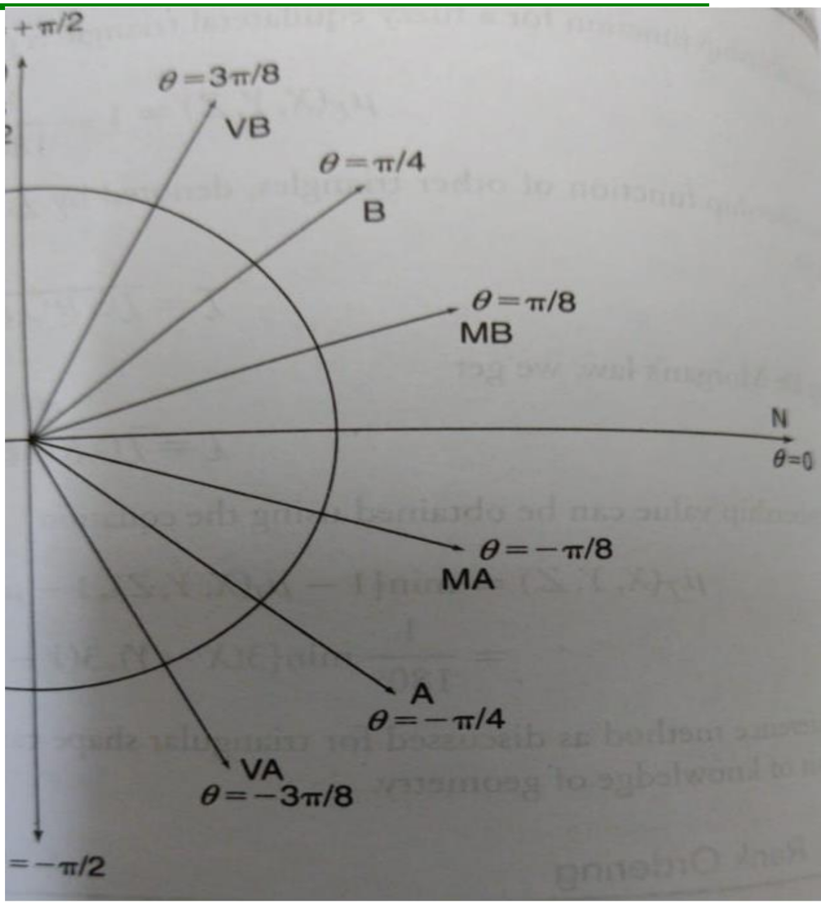
- Differ in coordinate description Defined on a universe of angles, hence they are of repeating shapes for every 2π cycles
- Angular fuzzy sets are used in the quantitative description of the linguistic variables, which are known as "truth values".
- For example, let's consider that pH values of water samples are taken from a contaminated pond. We know that,
- If pH=7, it is a neutral solution.
- Levels of pH between 14 and 7 are labeled as absolute basic (AB), very basic (VB), basic, fairly basic (FB), neutral (N) drawn from $\theta=\pi/2$ to $\theta=0$
- Levels of pH between 7 to 0 are called neutral, fairly acidic (FA), acidic (A), very acidic (VA), absolutely acidic (AA), are drawn from $\theta=0$ to $\theta=(-\pi/2)$.

Angular Fuzzy Sets

- pH values of water samples are taken from a contaminated pond
- If pH=7, then it is a neutral solution
- Levels of pH between 14 and 7 are labeled as absolute basic (AB), very basic (VB), basic, fairly basic (FB), neutral (N) drawn from $\theta=\pi/2$ to $\theta=0$
- Levels of pH between 7 to 0 are called neutral, fairly acidic (FA), acidic (A), very acidic (VA), absolutely acidic (AA), are drawn from $\theta=0$ to $\theta=(-\pi/2)$.

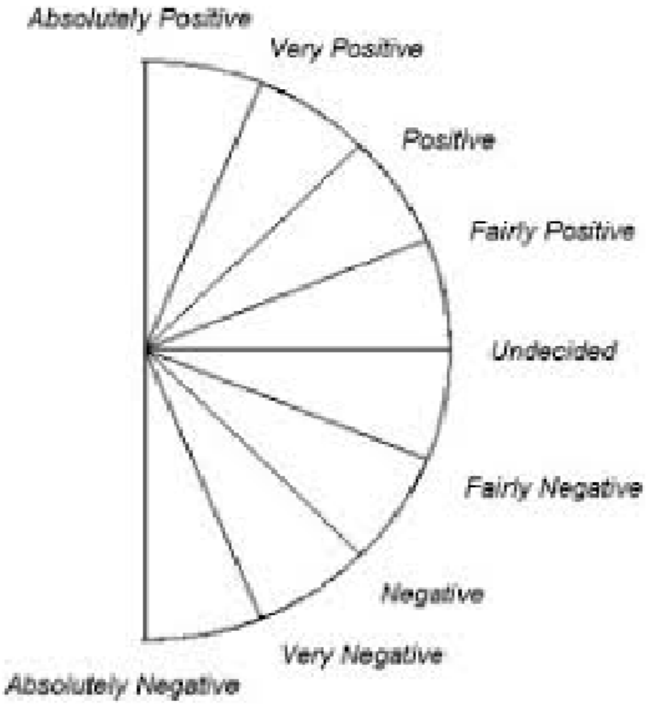
ANGULAR FUZZY EXAMPLE

- Linguistic values vary with θ and their membership values are given by the equation $\mu_t(\theta) = t \tan\theta$
- here ‘t’ is the horizontal projection of the radial vector



ANOTHER SIMPLE EXAMPLE

-



NEURAL NETWORKS

- Fuzzy membership functions using fuzzy classes of input data set
- Input values divided into training and testing set
- Weights of neurons are computed during training and can be verified using testing data

Genetic Algorithms

- Some membership functions and their shapes are assumed for various fuzzy variables
- These membership functions are the coded as bit strings and then concatenated
- A fitness function evaluates the goodness of each set of membership functions

INDUCTIVE REASONING

- Deriving general from specific
- Generates membership functions based on data provided
- Useful for complex systems where data is abundant and static
- Employs entropy minimization principle, which clusters the parameters of output classes
- Intent of induct is to discover a law having objective validity and universal application
- Inductive reasoning: derive the generic from the specific, leads to entropy minimization analysis that determines the quantity of information in a given data set
- Entropy is a metric that's a measure of the amount of disorder in a vector
- A situation is presented, you look at evidence from past similar situations and draw a conclusion based on the information available.

INDUCTIVE EXAMPLE

- A fuzzy threshold is established between classes of data
- Partitions are selected based on minimum entropy principle

LEVEL SET

- Crisp set consisting of the membership values of its singletons
- L(F)= $\left\{x \middle| 0 < x \leq 1 \; and \; \exists y \in U \; such \; that \; \mu_{F(y)} = x\right\}$

Example

- If U={2,3,4,5}& $A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$
- Find level set of A

- L(A)={1,0.5,0.3,0.2}

PROBLEM

- Let the maturity is computed based on age. If x is the age, membership in maturity

$$\mu_F(x) = \begin{cases} 0 & if \ x < 5 \\ \left(\dfrac{x-5}{20}\right)^2 & if \ 5 \le x \le 25 \\ 1 & if \ x > 25 \end{cases}$$

- If u={Sindu,Minul,Pearly,George,Omana,Lila} with ages 15,20,10,27,32,3 find normalcy, height, support, core and cardinality

SOLUTION

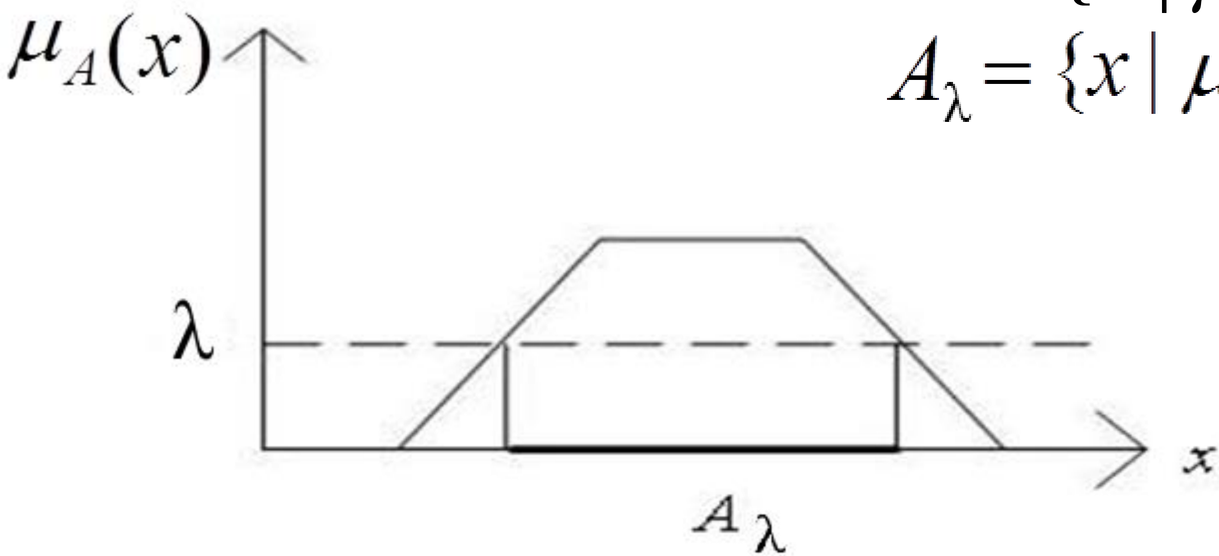
- Normalcy- Normal
- Height=1
- Support={Sindu,Minul,Pearly,George,Omana}
- Core= {George,Omana}
- Cardinality=0.25+0.5625+0.0625+1+1+0=2.8725

λ- CUT

λ-cut of a fuzzy set A_λ or (α-cut $^\alpha A$) is a crisp set A that contains all the elements in X that have membership value in A greater than or equal to λ (or α)

$$^{\alpha}A = \{x \mid \mu_A(x) \geq \alpha\}$$

$$A_\lambda = \{x \mid \mu_A(x) \geq \lambda\}$$



STRONG α-CUT

- A strong α-cut of a fuzzy set A is a crisp set ${}^{\alpha+}A$ that contains all the elements in X that have membership value in A strictly greater than α

$${}^{\alpha+}A = \{x \mid \mu_A(x) > \alpha\}$$

Example 1

- Find α-cut of A for α =0.5

$$A=\left\{\begin{aligned}&\frac{1}{1}+\frac{1}{2}+\frac{0.75}{3}+\frac{0.5}{4}+\\&\frac{0.3}{5}+\frac{0.3}{6}+\frac{0.1}{7}+\frac{0.1}{8}\end{aligned}\right\}$$

Find α cut for α =0.5

For α =0.5, $^{\alpha}A = \{1,2,3,4\}$

Example 2

Let $U=\{X_1, X_2, X_3, X_4, X_5\}$ and

$$A=\left\{\frac{0.6}{x_1}+\frac{0.5}{x_2}+\frac{0.3}{x_3}+\frac{0.7}{x_4}+\frac{1}{x_5}\right\}$$

Find α-cuts for 1, 0.7, 0.6, 0.2

$F_{1.0}=\{X_5\}$; $F_{0.7}=\{X_4, X_5\}$

$F_{0.6}=\{X_1, X_4, X_5\}$;

$F_{0.2}=\{X_1, X_2, X_3, X_4, X_5\}$;

Example 3

Fuzzy set A is defined on the universe X=[0, 5] with the membership function

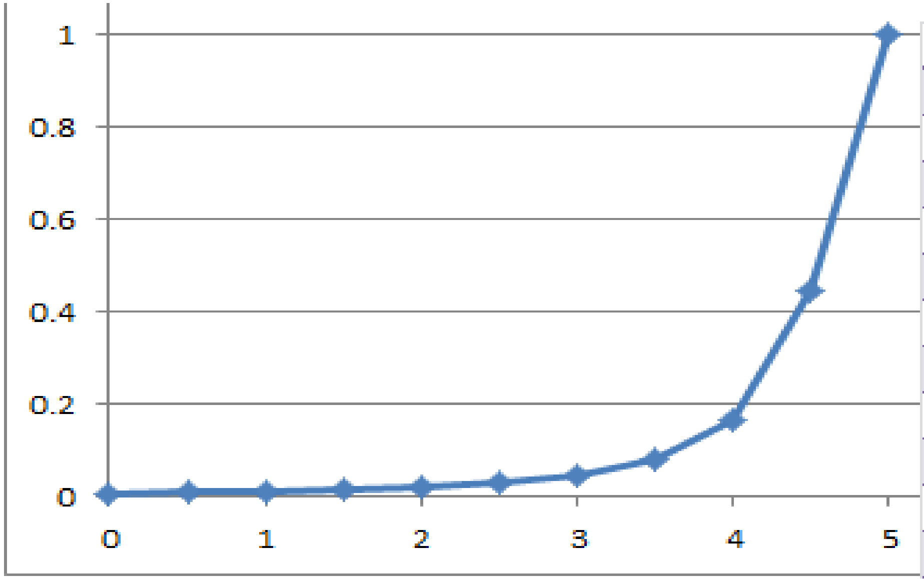
$$\mu_A(x) = \frac{1}{1 + 5(x - 5)^2}$$

Sketch membership function and define intervals along with x axis corresponding to α-cuts for the following values of α= 0.2,0.6,0.9 and 1.0

$A_{0.2}$ =[4.1,5]; $A_{0.6}$=[4.6,5]; $A_{0.9}$ =[4.85,5]; A_1 = [5,5]

Sketch of the Function

- . $$\mu_A(x) = \frac{1}{1+5(x-5)^2}$$



x	m
0	0.00793651
0.5	0.00977995
1	0.01234568
1.5	0.01606426
2	0.02173913
2.5	0.03100775
3	0.04761905
3.5	0.08163265
4	0.16666667
4.5	0.44444444
5	1

EXAMPLE 4

Fuzzy set A is defined on the universe X=[0, 5] with the membership function

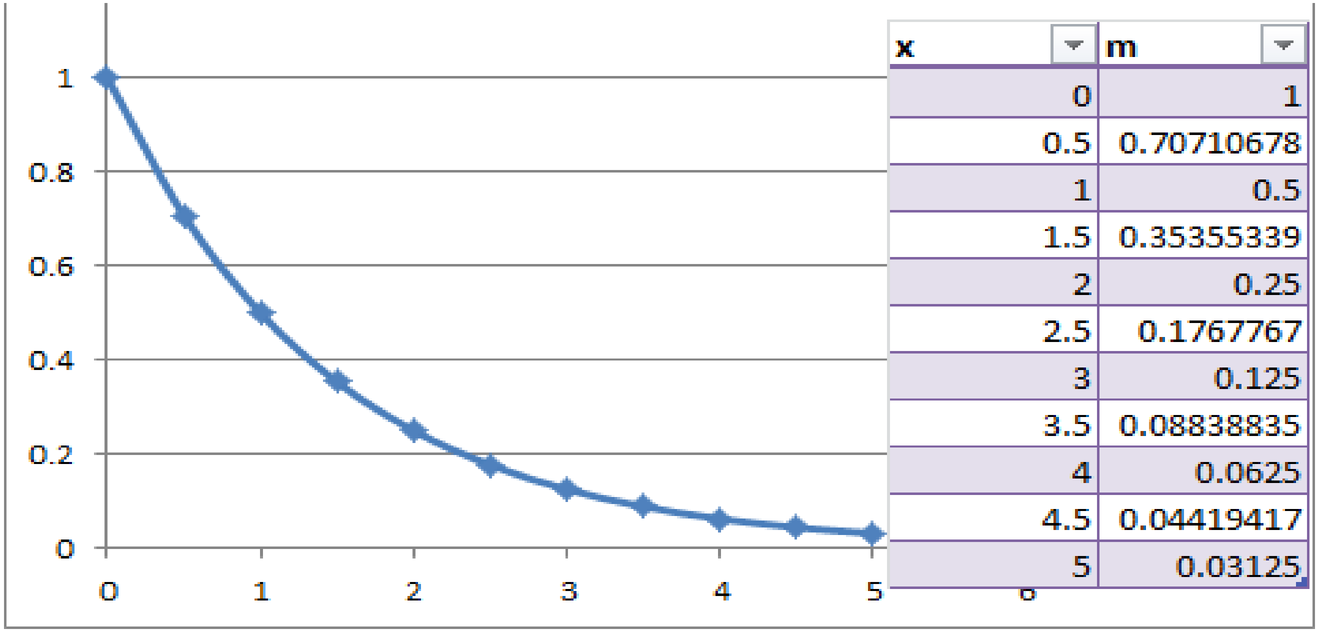
$$\mu_B(x) = 2^{-x}$$

Sketch membership function and define intervals along with x axis corresponding to α-cuts for the following values of α= 0.2,0.6,0.9 and 1.0

$A_{0.2}$ =[0,2.3]; $A_{0.6}$=[0,0.7]; $A_{0.9}$ =[0,0.15]; A_1 = [0,0]

SKETCH OF THE FUNCTION

- . $\mu_B(x) = 2^{-x}$



x	m
0	1
0.5	0.70710678
1	0.5
1.5	0.35355339
2	0.25
2.5	0.1767767
3	0.125
3.5	0.08838835
4	0.0625
4.5	0.04419417
5	0.03125

EXAMPLE 5

Determine the λ cut for the given fuzzy sets

$$S1 = \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100} \right\}$$

$$S2 = \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{0.95}{80} + \frac{1.0}{100} \right\}$$

Express the following using λ=0.5

$S1 \cup S2$, $S1 \cap S2$, $\overline{S1}$, $\overline{S1 \cap S2}$, $\overline{S1 \cup S2}$ for $\lambda = 0.5$

S1∪S2, S1∩S2, $S1$, $S2$, S1∩S2, S1∩S2

SOLUTION

$$S1 \cup S2 = \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100} \right\}$$

$(S1 \cup S2)_{0.5} = \{20,40,60,80,100\}$

$$S1 \cap S2 = \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{0.95}{80} + \frac{1.0}{100} \right\}$$

$((S1 \cap S2)_{0.5} = \{40,60,80,100\}$

$$\overline{S1} = \left\{ \frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} + \frac{0}{80} + \frac{0}{100} \right\}$$

$(\overline{S1})_{0.5} = \{0,20\}$

$$\overline{S1 \cap S2} = \left\{ \frac{1}{0} + \frac{0.55}{20} + \frac{0.4}{40} + \frac{0.2}{60} + \frac{0.05}{80} + \frac{0.0}{100} \right\}$$

$(\overline{S1 \cap S2})_{0.5} = \{0,20\}$

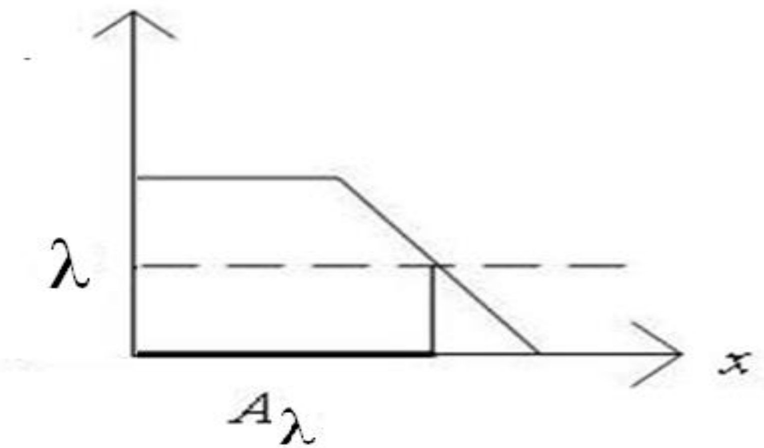
$$\overline{S1 \cup S2} = \left\{ \frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} + \frac{0.0}{80} + \frac{0.0}{100} \right\}$$

$(\overline{S1 \cup S2})_{0.5} = \{0,20\}$

α-CUT EXAMPLE PROBLEM

- Young

$$A_1(x) = \begin{cases} 1 & x \le 20 \\ (35-x)/15 & 20 < x < 35 \\ 0 & x \ge 35 \end{cases}$$



$$\frac{(35-x)}{15} = \lambda \qquad x = 35\text{-}15\lambda$$

$$A_{1_\lambda} = [0, 35 - 15\lambda] \ \forall \ \lambda \in (0,1]$$

EXAMPLE

- Middle Aged A_2 and Old A_3

$$A_2(x) = \begin{cases} 0 & x \le 20 \; or \quad x \ge 60 \\ (x-20)/15 & 20 < x < 35 \\ (60-x)/15 & 45 < x < 60 \\ 1 & 35 \le x \le 45 \end{cases}$$

$$A_3(x) = \begin{cases} 0 & x \le 45 \\ (x-45)/15 & 45 < x < 60 \\ 1 & x \ge 60 \end{cases}$$

α-CUT EXAMPLE

- Young, Middle-aged and Old

$$\left.\begin{aligned} {}^{\alpha}A_1 &= [0, 35-15\alpha] \\ {}^{\alpha}A_2 &= [15\alpha+20, 60-15\alpha] \\ {}^{\alpha}A_3 &= [15\alpha+45, 80] \end{aligned}\right\} \forall \alpha \in (0,1]$$

STRONG α-CUT EXAMPLE

- Young, Middle-aged and Old

$$\left.\begin{array}{r} {}^{\alpha+}A_1 = (0, 35 - 15\alpha) \\ {}^{\alpha+}A_2 = (15\alpha + 20, 60 - 15\alpha) \\ {}^{\alpha+}A_3 = (15\alpha + 45, 80) \end{array}\right\} \forall \alpha \in [0,1)$$

PROPERTIES

- $(A \cup B)_\lambda = (A_\lambda \cup B_\lambda)$
- $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$
- $(\bar{A})_\lambda \neq (\bar{A}_\lambda)$ except when $\lambda=0.5$
- For any $\lambda \leq \beta$, *where* $0 \leq \beta \leq 1$, it is true that $A_\beta \subseteq A_\lambda$ *where* $A_0 = X$

λ- Cut for Fuzzy Relations

- λ-cut of a fuzzy relation R_λ is defined on the universes X and Y that contains all pairs (x,y) in R that have membership value greater than or equal to λ

$$R_\lambda = \{(x,y) \mid \mu_R(x,y) \geq \lambda\}$$

Properties of Fuzzy Relations

- $(R \cup S)_\lambda = (R_\lambda \cup S_\lambda)$
- $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$
- $(\bar{R})_\lambda \neq \bar{R}_\lambda$ except when $\lambda=0.5$
- For any $\lambda \leq \beta, where\ 0 \leq \beta \leq 1$, it is true that $R_\beta \subseteq R_\lambda$

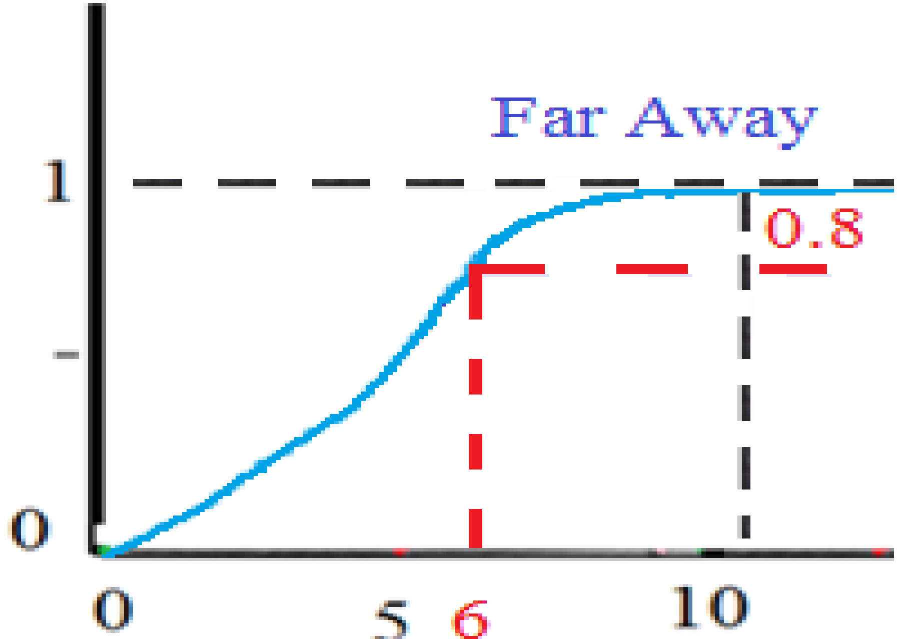
FUZZIFICATION

- Process of transforming a crisp input to fuzzy set
- Crisp quantities converted into linguistic variables
- If temperature is 20 degree Celsius, it is translated into cold, warm etc.

FUZZIFICATION

- Fuzzification of inputs determines the degree of matching of input with fuzzy sets

FUZZIFICATION EXAMPLE

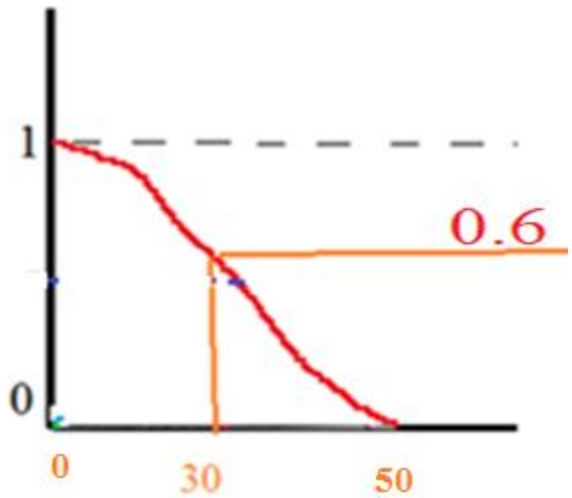
- Fuzzification of distance of car wrt far-away fuzzy set
- A car is found at a distance of 6 m
- 

Result of fuzzification

Input- Distance of the car(0-10 range)- 6

MATCHING – ANOTHER EXAMPLE

- Speed of the Car is 30 km/hour- to fuzzy set **Slowly**



PROCESSING

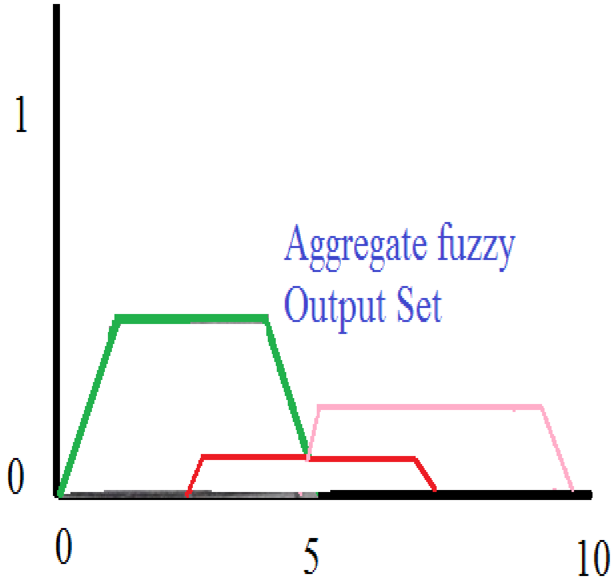
- Membership value corresponding to input is computed
- Output fuzzy set is reshaped based on input matching
- Using membership value assignment methods

DEFUZZIFICATION

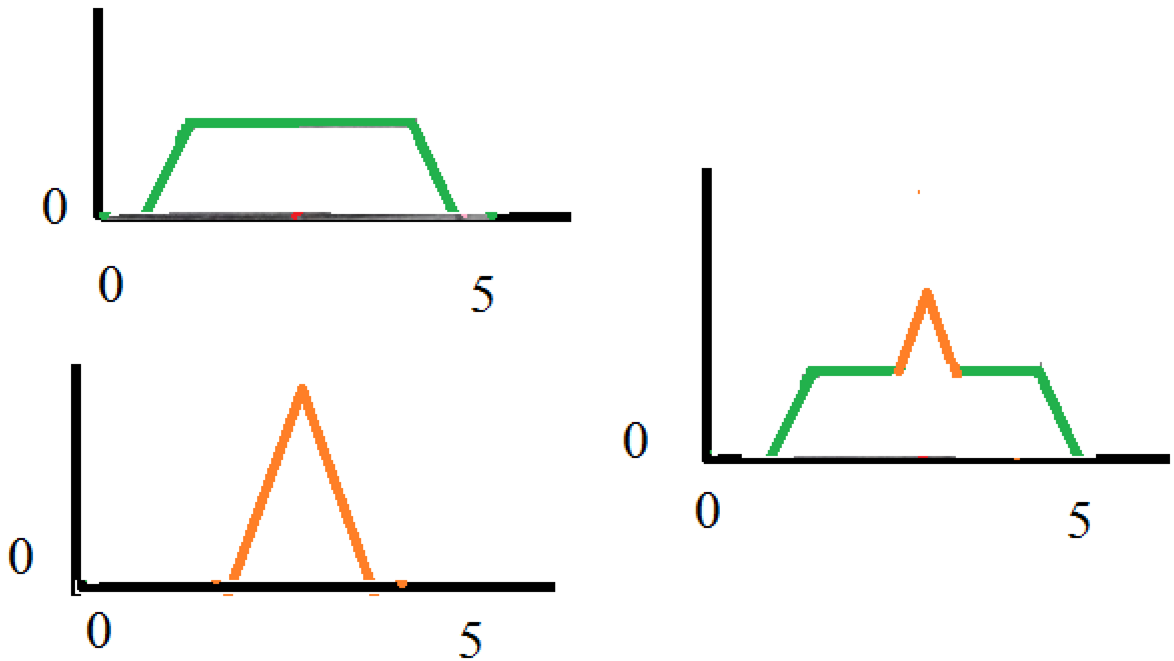
- Process of converting the fuzzy output sets into one crisp value for each output variable
- Output may be union of two or more fuzzy membership functions on the universe of discourse of the output variable

AGGREGATION OF OUTPUT

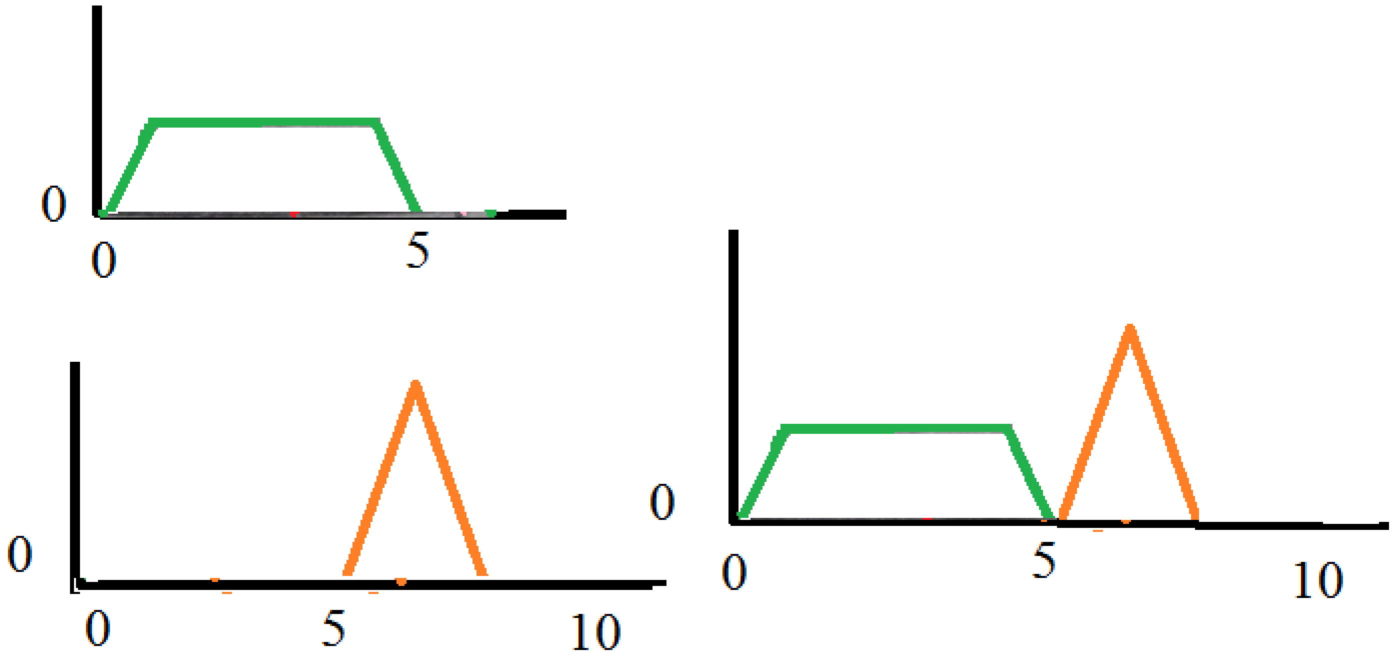
- Need to combine the effects of different rules that are applicable
- Different aggregation methods
  - Maximum among all
  - Algebraic sum of all



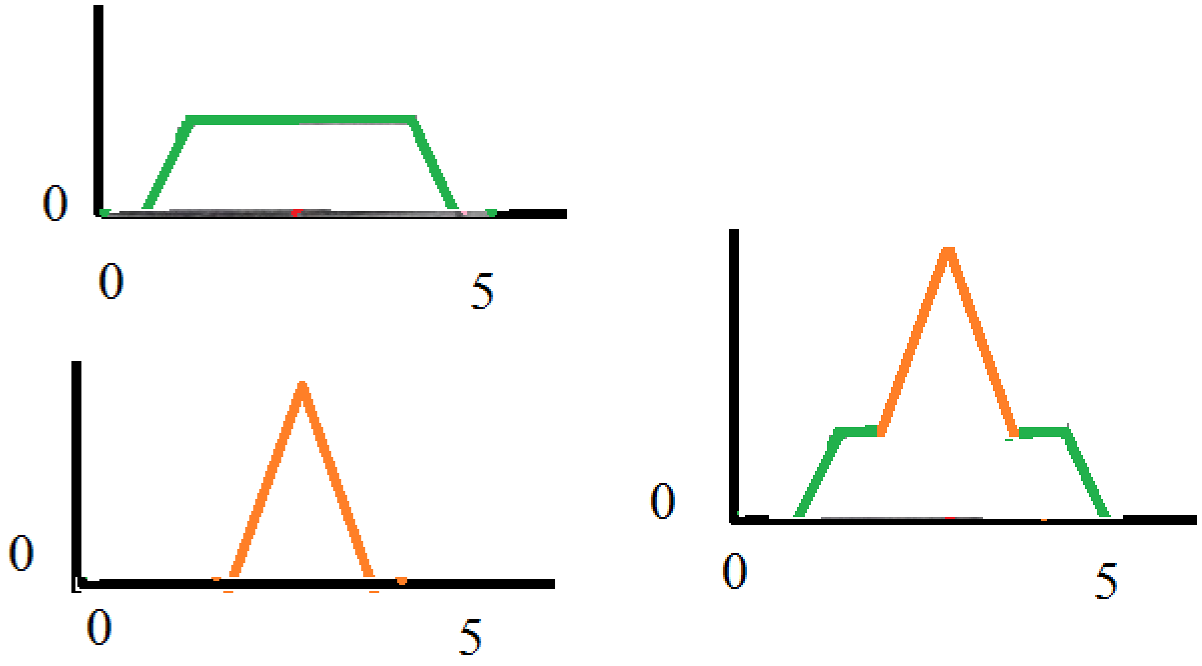
MAXIMUM



AGGREGATION EXAMPLE1



AGGREGATION EXAMPLE2



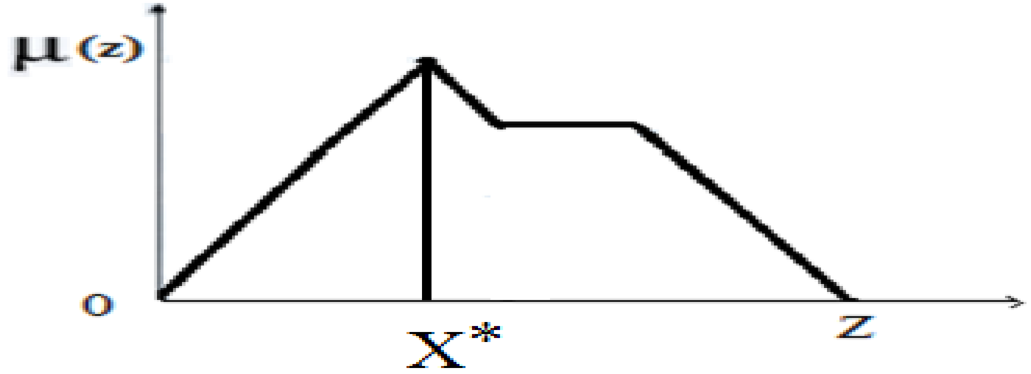
Defuzzification Methods

- Maximum- membership Principle
- Centroid method
- Weighted average method
- Mean of Max
- Centre of Sums
- Centre of largest area
- First of maxima, Last of maxima

MAX-MEMBERSHIP PRINCIPLE

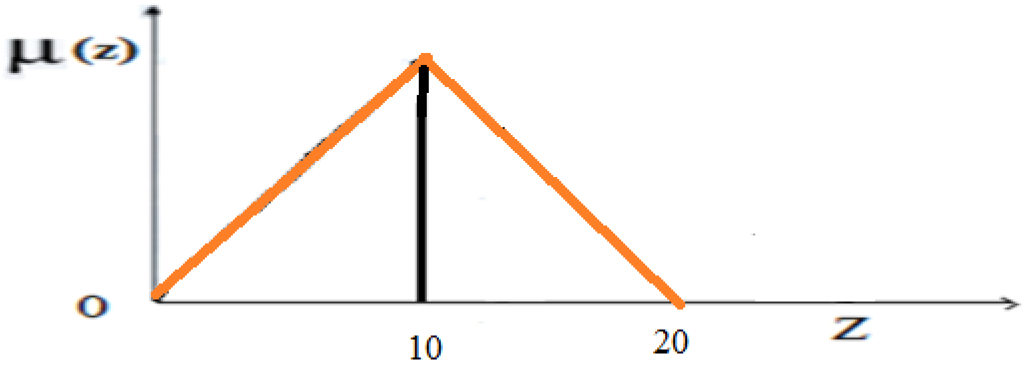
- Maximum membership value of the output function is chosen

$$\mu_c(x^*) \geq \mu_c(x) \quad \text{for all } x \in X$$



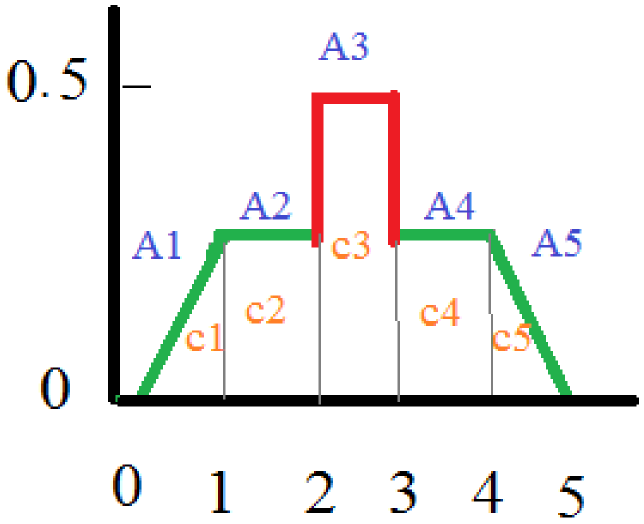
EXAMPLE

- Find the crisp value corresponding to the following output membership function using maximum membership



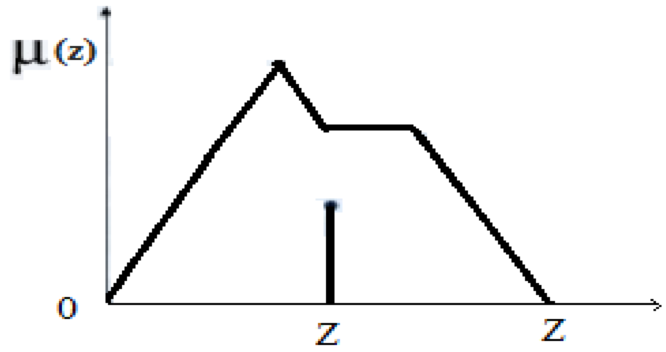
CENTROID METHOD

- Center of mass or center of area or center of gravity
- Individual output fuzzy sets are super imposed into a single aggregate fuzzy set using max method
- Defuzzified value is computed as the Centroid of the resultant



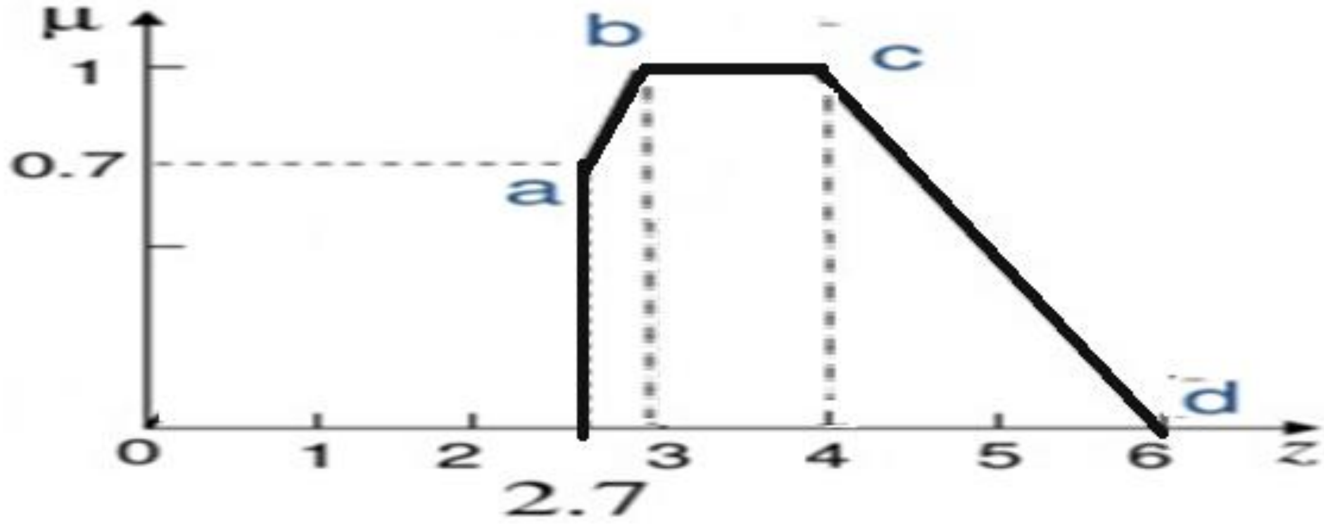
CENTROID - DEFUZZIFICATION

$$z^* = \frac{\int \mu_C(z) z dz}{\int \mu_C(z) dz}, \quad \text{for all } z \in Z$$



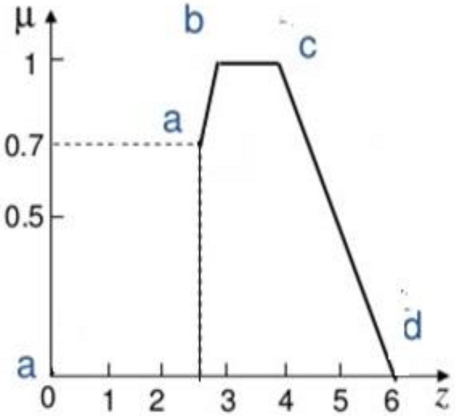
EXAMPLE

- Find the crisp value for the following using centroid



SOLUTION

- Equations



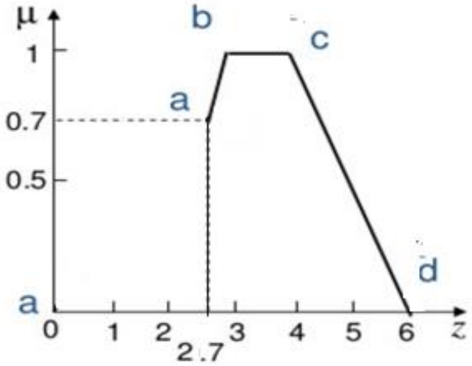
$$\mu(z) = \begin{cases} z-2 & 2.7 \le z < 3 \\ 1 & 3 \le z < 4 \\ -0.5z+3 & 4 \le z \le 6 \end{cases}$$

$$\mu(z) = \quad z-2 \quad 2.7 \le z < 3$$

$$\mu(z) = \quad 1 \quad 3 \le z < 4$$

$$\mu(z) = \quad -0.5z+3 \quad 4 \le z \le 6$$

SOLUTION



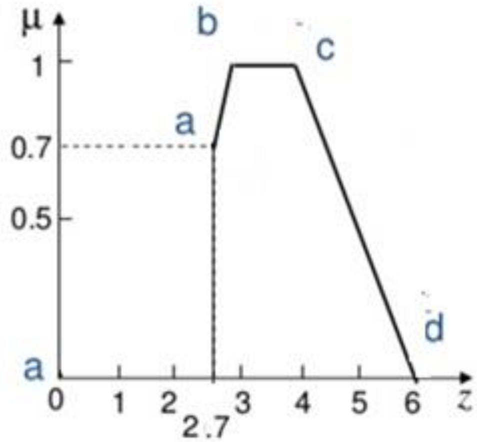
$$z^* = \frac{\int \mu_C(z) z dz}{\int \mu_C(z) dz}, \quad \text{for all } z \in Z$$

$$\mu(z) = \begin{cases} z-2 & 2.7 \le z < 3 \\ 1 & 3 \le z < 4 \\ -0.5z+3 & 4 \le z \le 6 \end{cases}$$

$$\text{Numerator} = \int_{2.7}^{3} (z^2 - 2z) dz + \int_{3}^{4} z\, dz + \int_{4}^{6} (-0.5z^2 + 3z) dz$$

$$\text{Denominator} = \int_{2.7}^{3} (z-2) dz + \int_{3}^{4} dz + \int_{4}^{6} (-0.5z+3) dz$$

SOLUTION



$$\mu(z) = \begin{cases} z-2 & 2.7 \le z < 3 \\ 1 & 3 \le z < 4 \\ -0.5z+3 & 4 \le z \le 6 \end{cases}$$

$$\text{Numerator} = \int_{2.7}^{3} (z^2 - 2z)\,dz + \int_{3}^{4} z\,dz + \int_{4}^{6} (-0.5z^2 + 3z)\,dz$$

$$\text{Denominator} = \int_{2.7}^{3} (z^1 - 2)\,dz + \int_{3}^{4} dz + \int_{4}^{6} (-0.5z + 3)\,dz$$

NUMERATOR

$$= \left[\frac{z^3}{3} - z^2 \right]_{2.7}^{3} + \left[\frac{z^2}{2} \right]_{3}^{4} + \left[\frac{-0.5 * z^3}{3} - \frac{z^2}{2} \right]_{4}^{6}$$

$$= 2.44 - 1.71 + 3.5 - 25.33 + 30 = 8.9$$

DENOMINATOR

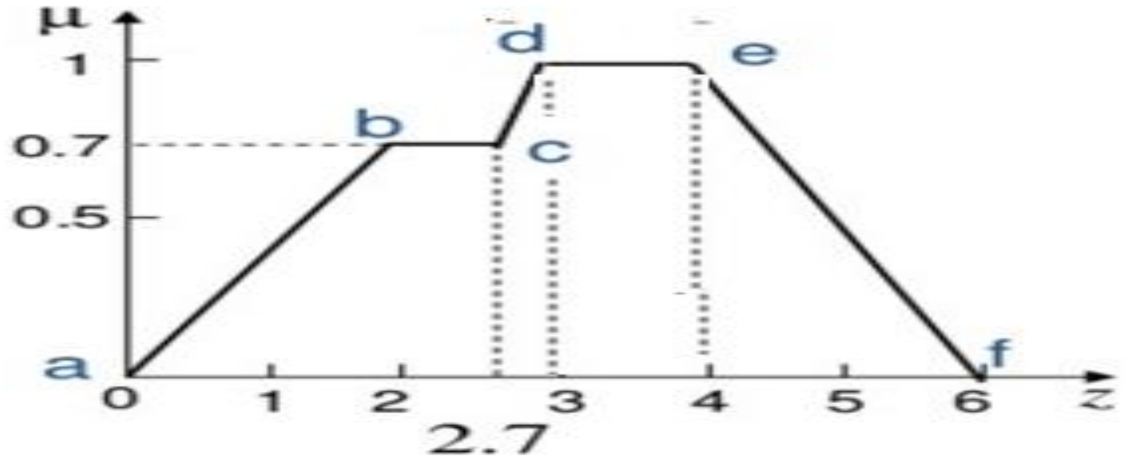
$$= \left[\frac{Z^2}{2} - 2Z \right]_{2.7}^{3} + \left[Z \right]_{3}^{4} + \left[\frac{-0.5 * Z^2}{2} - 3Z \right]_{4}^{6}$$

$$= 0.25 + 1 + 1 = 2.25$$

$$\text{Output} = 8.9 / 2.25 = 3.9$$

EXAMPLE

- Find the crisp value for the following using centroid



SOLUTION

- Equations

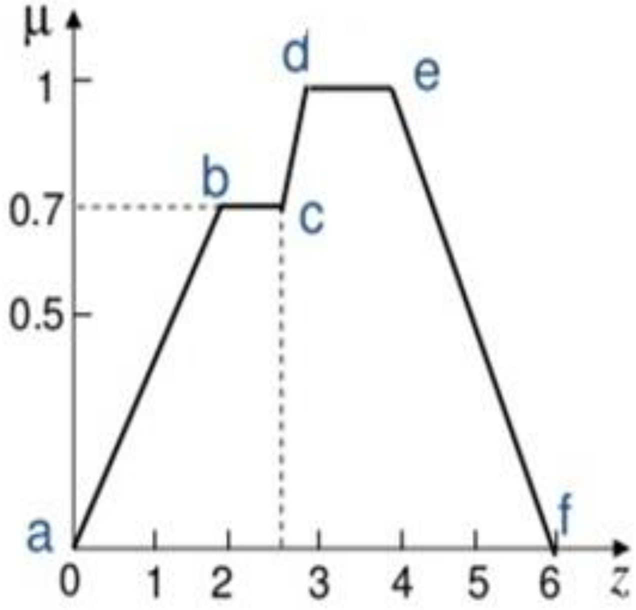
$$\mu(z) = 0.35z \qquad 0 \le z < 2$$

$$\mu(z) = 0.7 \qquad 2 \le z < 2.7$$

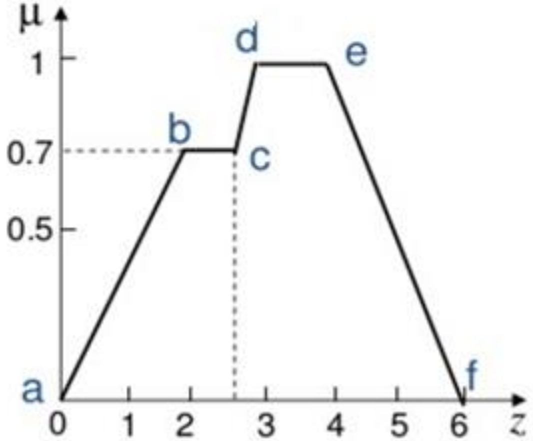
$$\mu(z) = z - 2 \qquad 2.7 \le z < 3$$

$$\mu(z) = 1 \qquad 3 \le z < 4$$

$$\mu(z) = -0.5z + 3 \qquad 4 \le z \le 6$$



Example- Solved



$$z^* = \frac{\int \mu_{\tilde{C}}(z)\, z\, dz}{\int \mu_{\tilde{C}}(z)\, dz}$$

$$\text{Numerator} = \int_0^2 0.35 z^2\, dz + \int_2^{2.7} 0.7 z\, dz + \int_{2.7}^{3} (z^2 - 2z)\, dz + \int_3^4 z\, dz + \int_4^6 (-0.5 z^2 + 3z)\, dz$$

$$\mu(z) = \begin{cases} 0.35z & 0 \le z < 2 \\ 0.7 & 2 \le z < 2.7 \\ z - 2 & 2.7 \le z < 3 \\ 1 & 3 \le z < 4 \\ -0.5z + 3 & 4 \le z \le 6 \end{cases}$$

$$\text{Denominator} = \int_0^2 0.35z\, dz + \int_2^{2.7} 0.7\, dz + \int_{2.7}^{3} (z - 2)\, dz + \int_3^4 dz + \int_4^6 (-0.5z + 3)\, dz$$

NUMERATOR

$$= \left[\frac{0.35 * Z^3}{3} \right]_0^2 + \left[0.35 * Z^2 \right]_2^{2.7} + \left[\frac{Z^3}{3} - Z^2 \right]_{2.7}^3$$

$$+ \left[\frac{Z^2}{2} \right]_3^4 + \left[\frac{-0.5 * Z^3}{3} - \frac{Z^2}{2} \right]_4^6$$

$$=0.93+1.15+2.44-1.71+3.5-25.33+30=10.98$$

DENOMINATOR

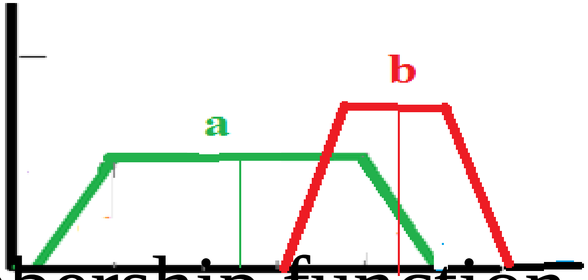
$$= \left[\frac{0.35 * Z^2}{2}\right]_0^2 + \left[0.7 * Z\right]_2^{2.7} + \left[\frac{Z^2}{2} - 2Z\right]_{2.7}^3 + [Z]_3^4 + \left[\frac{-0.5 * Z^2}{2} - 3Z\right]_4^6$$

$$=0.7+ \quad 0.49+ \quad 0.25+ \quad 1 +1=3.44$$

$$\text{Output} = 10.98 / 3.44 = 3.19$$

WEIGHTED AVERAGE METHOD

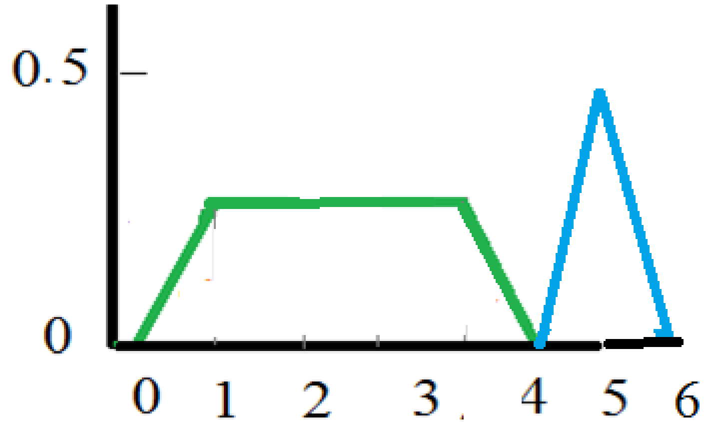
- Each membership function is weighted by its maximum membership value
- Overlapping areas counted multiple times
- $X^* = \dfrac{\sum \mu_c(xi').xi'}{\sum \mu_c(xi')}$



- xi' is the maximum of the ith membership function
- For a symmetrical function we are using this .
- Maximum membership point can be determined from the mean of the values

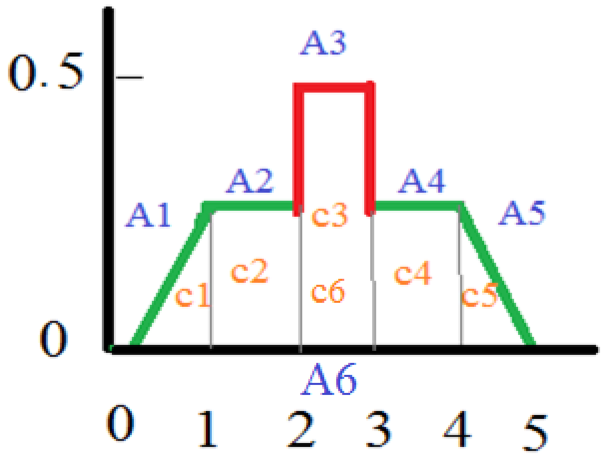
EXAMPLE

- $weighted\ average$
- $X^* = \frac{2*0.3+5*0.5}{0.3+0.5} = 3.9$

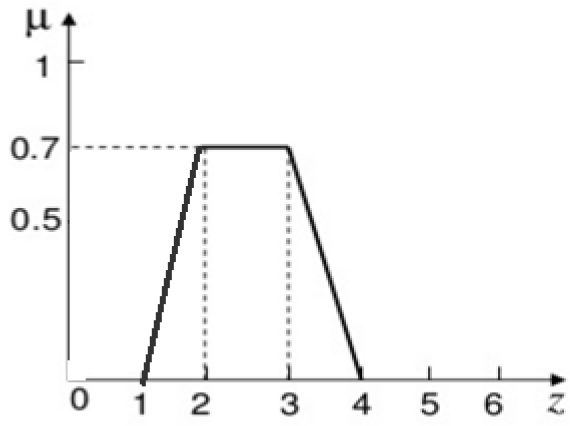


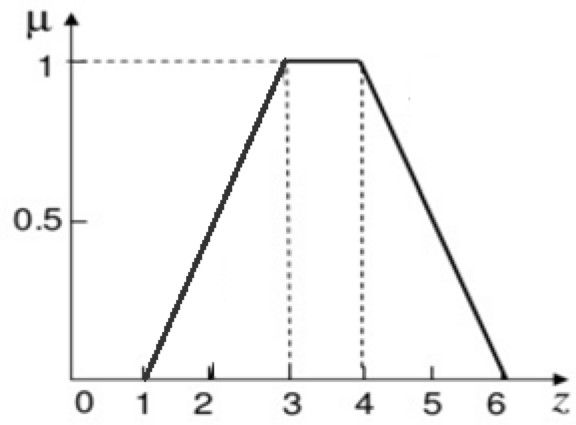
CENTER-OF- SUMS METHOD

- Similar to centroid method; but aggregate output set is obtained by sum method
- Overlapping areas are counted multiple times
- For discrete fuzzy sets
- $X^* = \frac{\sum_{i=1}^{n} xiAi}{\sum_{i=1}^{n} Ai}$
- xi=center of the membership function



Center-of- Sums Method





- denomenator= [.5*(1+3)*.7]+[0.5*(1+5)*1=4.4
- numerator= 2.5*[.5*(1+3)*.7]+3.5*[0.5*(1+5)*1=14
- X*=14/4.4=3.18

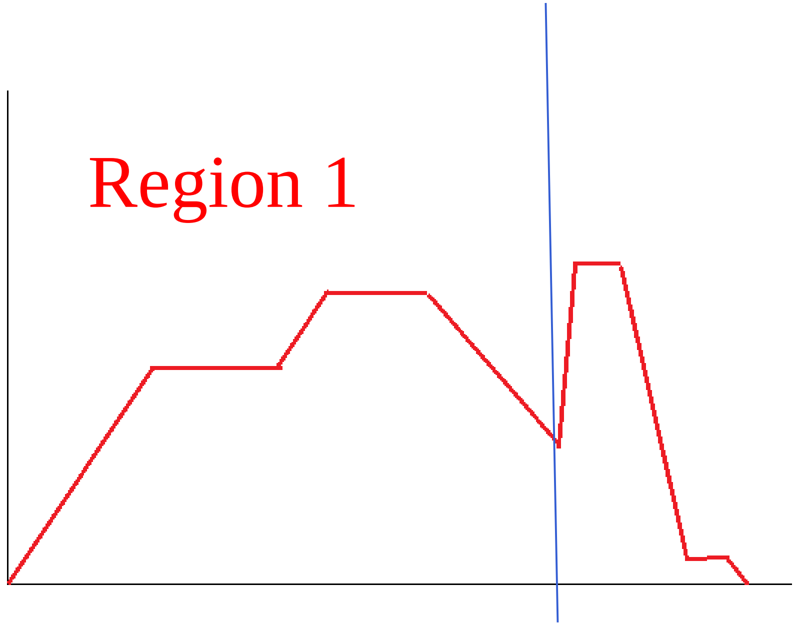
CENTRE OF LARGEST AREA

- Can be adopted when the output consists of two non overlapping convex sub regions
  - Monotonically increasing or decreasing or increasing and then decreasing with increasing values
- Center of gravity of largest region is used to obtain the defuzzified value

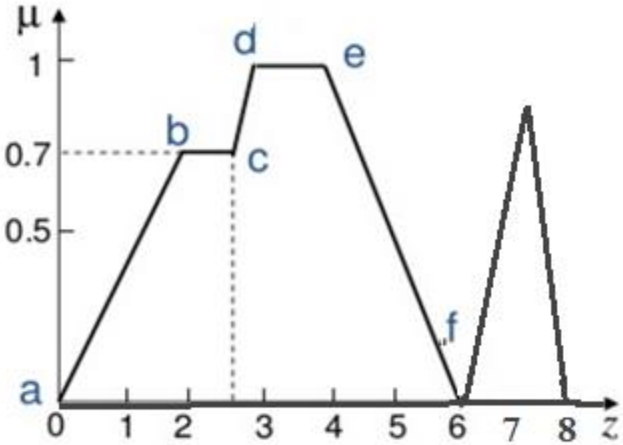
$$z^* = \frac{\int \mu_{Cj}(z) z dz}{\int \mu_{Cj}(z) dz}$$

EXAMPLE

- Region 1 is selected as the largest area



EXAMPLE



$$z^* = \frac{\int \mu_{\tilde{C}}(z)\, z\, dz}{\int \mu_{\tilde{C}}(z)\, dz}$$

$$\text{Numerator} = \int_0^2 0.35z^2\,dz + \int_2^{2.7} 0.7z\,dz + \int_{2.7}^{3} (z^2 - 2z)\,dz + \int_3^4 z\,dz + \int_4^6 (-0.5z^2 + 3z)\,dz$$

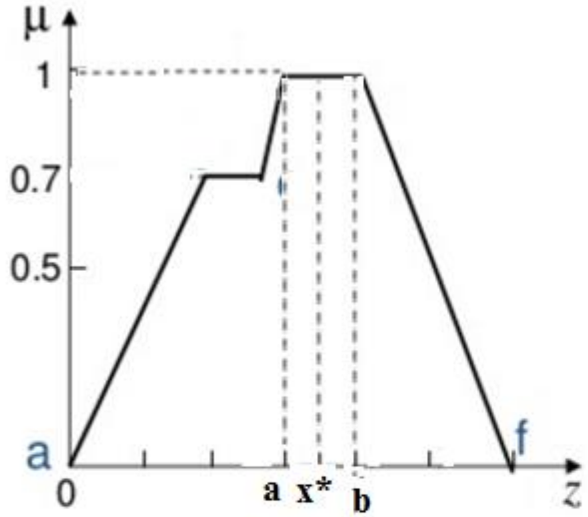
$$\mu(z) = \begin{cases} 0.35z & 0 \le z < 2 \\ 0.7 & 2 \le z < 2.7 \\ z-2 & 2.7 \le z < 3 \\ 1 & 3 \le z < 4 \\ -0.5z+3 & 4 \le z \le 6 \end{cases}$$

$$\text{Denominator} = \int_0^2 0.35z\,dz + \int_2^{2.7} 0.7\,dz + \int_{2.7}^{3} (z-2)\,dz + \int_3^4 dz + \int_4^6 (-0.5z+3)\,dz$$

MEAN-OF-MAXIMA

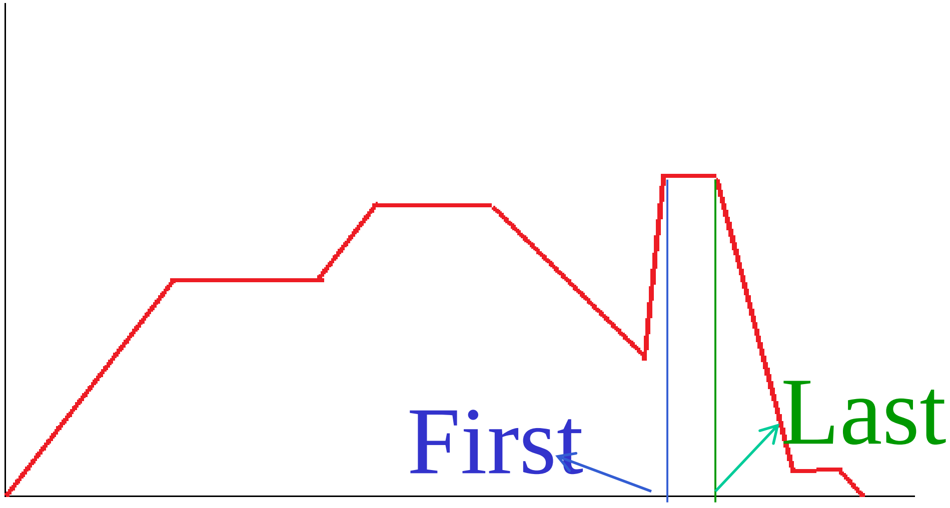
- Middle of maxima
- Highest degree of membership is returned
- If there are many elements x1, same highest membership

$$C = \frac{\sum_{i=1}^{k} xi}{k}$$ for discrete fuzzy set



First of Maxima/ Last

- Determines smallest value of the domain of aggregate membership function having maximum membership degree
- Similarly largest value for last



THANK YOU

MODULE 3

FUZZY DESCRIPTIONS

- Used by humans for expert tasks
- Fuzzy Variables
  - Qualitative with linguistic terms
  - Quantitative with MF
- Fuzzy if then rules
  - Transforms fuzzy input into fuzzy output
- Example
  - Temperature ranges and power requirement
  - If low temperature then high power required

LINGUISTIC VARIABLES

- Rules expressed using linguistic variables
- Words or sentences in natural language
- Terms characterized as atoms
- Approximate characterization of a complex problem
  - Name of variable, universe of discourse, fuzzy set
  - Syntactic rule for generating values and semantic rule for assigning meaning
  - Example-Old, Tall, Cold, young, slow, medium
  - Composite terms- somewhat old, very slow horse, fairly beautiful painting

- A linguistic variable is a fuzzy variable.
  - The linguistic variable speed ranges between 0 and 300 km/h and includes the fuzzy sets slow, very slow, fast, …
  - Fuzzy sets define the linguistic values.
- Hedges are qualifiers of a linguistic variable.
  - All purpose: very, quite, extremely
  - Probability: likely, unlikely
  - Quantifiers: most, several, few
  - Possibilities: almost impossible, quite possible

LINGUISTIC HEDGES

- Modifiers that change meaning of variable slightly
- Singular meaning of an atomic item is hedged or modified from its original interpretation
- Fundamental atomic terms such as tall, old can be modified with adjectives or adverbs like very as very tall
- Others- low, light, more or less, almost, nearly, slightly, fairly, mostly, roughly, Highly, moderately, plus, minus, rather

IMPLEMENTATION

- In fuzzy sets, these linguistic hedges have the effect of modifying the membership function of a basic term
- If α is a basic linguistic atom and $\mu_A(x)$ is the corresponding membership function, linguistic hedges such as α^2, α^4, $\alpha^{1.25}$ can be used to modify basic atom
- known as concentrations, dilations, intensification

CONCENTRATION EXAMPLES

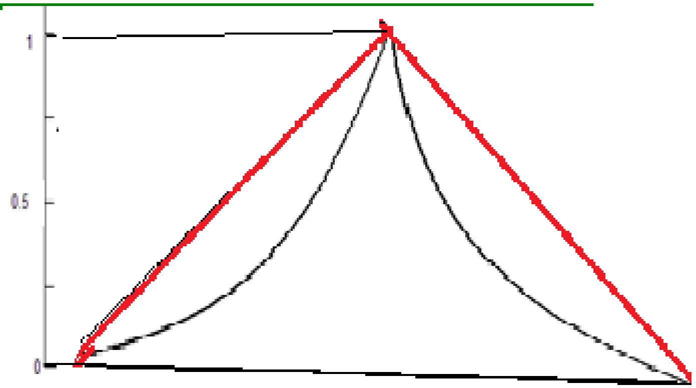
- Tend to concentrate the elements of a fuzzy set by reducing the degree of a membership that are partly in the set
- Very $\alpha = \alpha^2$
- Very very $\alpha = \alpha^4$
- Plus $\alpha = \alpha^{1.25}$

Dilation Examples

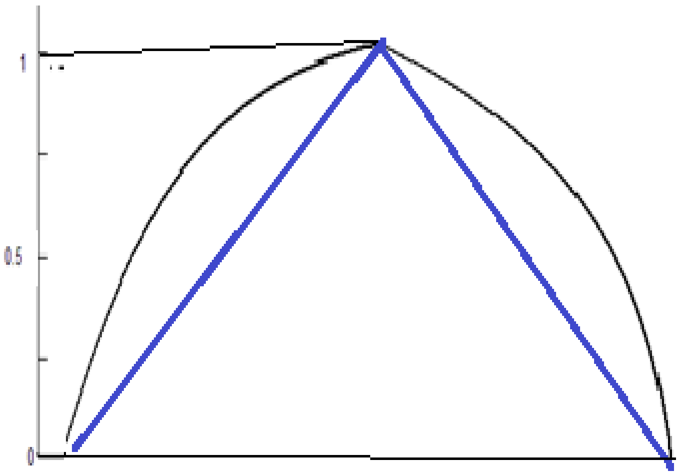
- Stretch or dilate a fuzzy set by increasing the membership value of all elements that are partly in the set
- Slightly $\alpha = \sqrt{\alpha}$
- Minus $\alpha = \alpha^{0.75}$

GRAPHICAL REPRESENTATION

- Concentration

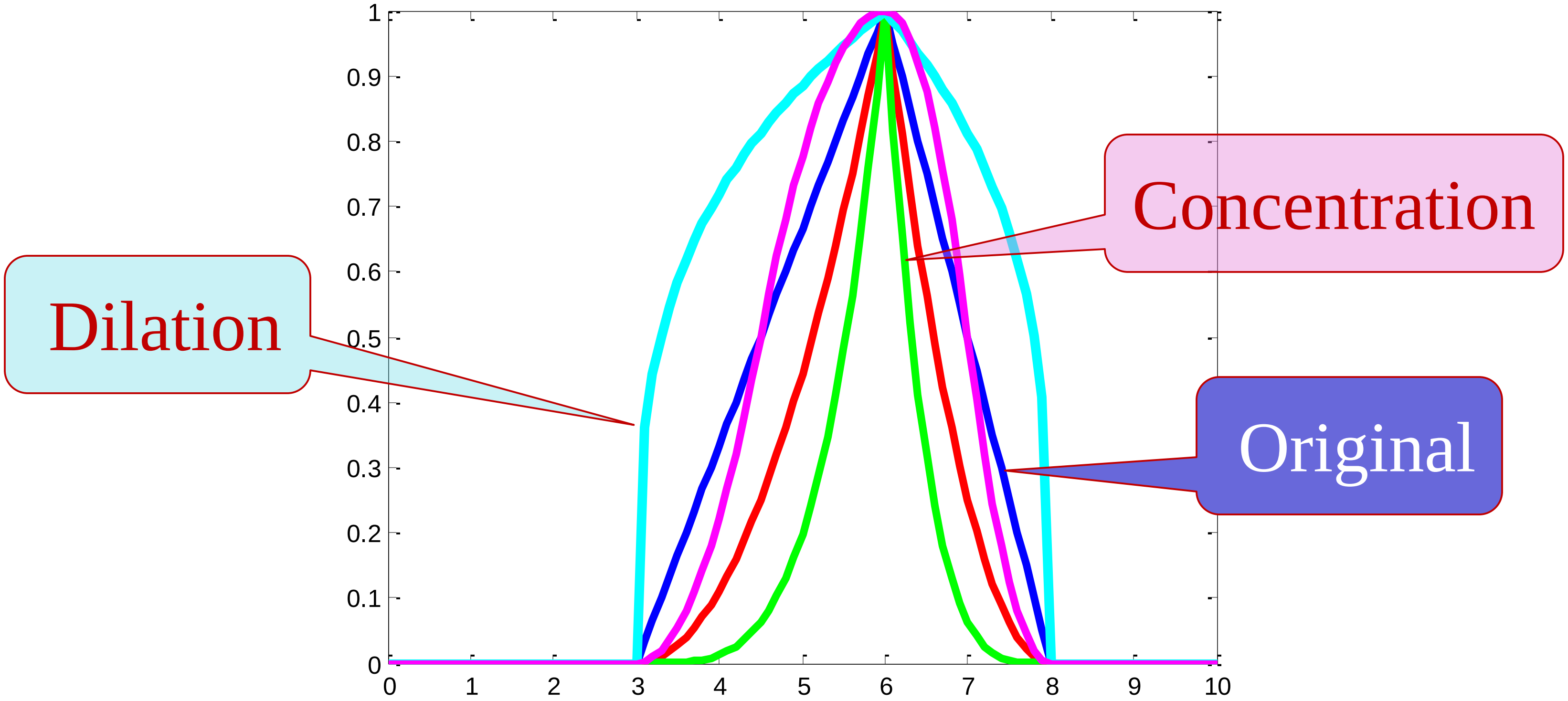


- Dilation



DIFFERENT OPERATIONS

- MATLAB implementation output



PROBLEM1

- Find Very small, not very small, not very large, not very very large

$$\text{Small} = \left\{ \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5} \right\}$$

$$\text{Large} = \left\{ \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} \right\}$$

Very Small

$$\text{Small} = \left\{ \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5} \right\}$$

$$\text{Very Small} = \text{Small}^2$$
$$= \left\{ \frac{1}{1} + \frac{0.64}{2} + \frac{0.36}{3} + \frac{0.16}{4} + \frac{0.04}{5} \right\}$$

NOT VERY SMALL

$$\text{Small} = \left\{ \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5} \right\}$$

$$\text{Not Very Small} = 1 - \text{Small}^2$$

$$= \left\{ \frac{0}{1} + \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5} \right\}$$

NOT VERY LARGE

$$\text{Large} = \left\{ \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} \right\}$$

$$\text{NotVeryLarge} = \left\{ \frac{.96}{1} + \frac{.84}{2} + \frac{0.64}{3} + \frac{0.36}{4} \right\}$$

NOT VERY VERY LARGE

$$\text{Large} = \left\{ \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} \right\}$$

$$= \left\{ \frac{1}{1} + \frac{1}{2} + \frac{0.9}{3} + \frac{0.6}{4} \right\}$$

PROBLEM 2

- Find almost small

0.894427
0.774597
0.632456
0.447214

$$\text{Small} = \left\{ \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5} \right\}$$

$$\text{AlmostSmall} = SQRT(small) = \left\{ \frac{1}{1} + \frac{0.89}{2} + \frac{0.77}{3} + \frac{0.63}{4} + \frac{0.45}{5} \right\}$$

PROPOSITIONS

- Text sentences expressed in any language
- Is a declarative sentence that is either T or F
- Canonical form
  - z is P- z symbol, P predicate
- Subject- what (or whom) the sentence is about
- Predicate-part of a sentence, or a clause, that tells what the subject is doing or what the subject is
  - Kothamangalam is in Kerala
  - Kothamangalam – subject, in Kerala- predicate
- Every proposition has its opposite- negation

BINARY OPERATIONS

- Truth tables define logic functions of two propositions
- Let X and Y be two propositions, either of which can be true or false
- Conjunction- X and Y-
- Disjunction- X or Y
- Implication- If X then Y
- Equivalence- X if and only if Y

INFERENCE RULES

- Can be formulated based on the operations

$$(x \wedge (x \rightarrow y)) \rightarrow y$$

$$(\bar{y} \wedge (x \rightarrow y)) \rightarrow \bar{x}$$

$$((x \rightarrow y) \wedge (y \rightarrow z)) \rightarrow (x \rightarrow z)$$

- Certain propositions always true irrespective of x and y- tautologies

FUZZY TRUTH VALUES

- Truth value of a proposition “Z is A” or truth value of A, denoted tv(A) is in [0,1]
- Can be related to fuzzy sets by equating fuzzy truth values to degrees of membership to fuzzy sets
- Fuzzy set in [0,1] –linguistic truth value

RESULT OF OPERATIONS

- Truth value of a proposition can be obtained from logic operations of other propositions whose truth values are known
- Tv(X AND Y)=min{tv(X),tv(Y)}
- Tv(X OR Y)=max{tv(X),tv(Y)}
- Tv(NOT X)= 1-tv(X)
- Tv(X→Y)=max{1-tv(X), min[tv(X),tv(Y)]}

PROBLEM 3

- Find not Very Small and Not Very Very Large

$$\text{Not Very Small} = \left\{ \frac{0}{1} + \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5} \right\}$$

$$\text{Not Very Very Large} = \left\{ \frac{1}{1} + \frac{1}{2} + \frac{0.9}{3} + \frac{0.6}{4} \right\}$$

$$\text{Solution} = \left\{ \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.6}{4} \right\}$$

Tv(3 is not Very Small and Not Very Very Large)

0.64

Components- Propositions

- 1. Fuzzy predicates- fuzzy like tall
- 2. Fuzzy modifiers-very, fairly, moderately, rather
- 3. Fuzzy quantifiers- provides an imprecise characterization of the cardinality of one more sets
- Quantifiers – most, many, several, frequently
- Represent meaning of propositions contain probabilities
  - Eg-many people are educated

COMPONENTS- PROPOSITIONS-CONTD..

- 4. Fuzzy qualifiers
- Fuzzy qualifiers-4 modes of qualification
- Fuzzy truth qualification

It is represented as “x is τ”

where τ is a fuzzy truth value.

- Claims the degree of truth of a fuzzy proposition
- (Question paper is easy) is not very true

Qualified preposition

Qualifying fuzzy truth value

Components- Propositions-Contd..

- Fuzzy probability qualification
- x is λ

λ –fuzzy probability

Fuzzy probability is expressed by likely, very likely, unlikely , around and so on

  - Fuzzy probability is expressed
  - (Question paper is easy) is likely

COMPONENTS- PROPOSITIONS-CONTD..

- Fuzzy possibility qualification

x is π

π- fuzzy possibility

Possible Values can be interpreted as labels of fuzzy subsets

Quite possible, almost possible , almost impossible

(Question paper is easy) is almost impossible

COMPONENTS- PROPOSITIONS-CONTD..

- Fuzzy usuality qualification
- Usually (x)=Usually(X is F)
- Here subject X is a variable taking values in a universe of discourse U
- Predicate F is a fuzzy subset of U
- *usuality* propositions which are usually true or, events which have a high probability of occurrence

USUALITY PROPOSITIONS

- Expressed in the form *usually* (X is F), in which X is a variable taking values in a universe of discourse U and F is a fuzzy subset of U which may be interpreted as a *usual value* of X
- Examples-
- *Usually Mini is very cheerful*
- *Usually a TV set weighs about twenty kilograms*

FORMATION OF RULES

- In the field of AI , there are ways to represent knowledge. The most common way to represent human knowledge is to form it into natural language expressions
- IF (antecedent) then (consequent)

FUZZY IMPLICATION

- A fuzzy implication (also known as fuzzy If-then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

If x ***is*** A **then** y ***is*** B

where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y, respectively.

Often, x ***is*** A is called the **antecedent** or premise, while y ***is*** B is called the **consequence** or conclusion.

FUZZY IMPLICATION

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$

Fuzzy IF THEN Rules

- Formulated using Fuzzy sets and relations
- If a set of conditions are satisfied, a set of consequences can be inferred
- Fuzzy rule R:if 'x is A' Then 'y is B' is expressed as R: $A(x) \rightarrow B(y)$
- Can be expressed as a fuzzy relation between A and B where
- $R(x,y) = \tau[A(x) \rightarrow B(y)]$

TYPES OF STATEMENTS

- Assignment
  - Temperature=high, He is old
- Conditional Statements
  - IF temperature is high THEN close the valve
  - IF y is very cool THEN stop
- Unconditional Statements
  - Open the valve
  - Goto sum
  - Turn the pressure low
- Both conditional and unconditional statements impose some restrictions on the consequent

Compound Rules

- Compound rule is collection of many simple rules combine together
- Any compound rule structures may be decomposed into a series of canonical simple rules
- Rules are general based on natural language representation
- Different Types
  - Multiple conjunctive antecedents
  - Multiple disjunctive antecedents
  - Conditional statements with else and unless
  - Nested IF-THEN

Multiple conjunctive antecedents

- Example-IF X is A1,A2, …, An THEN Y is Bm
- To perform the Decomposition
- Assume a fuzzy set

$$Am = A_1 \cap A_2 \cap .... \cap An$$

- Expressed using the membership function
- $\mu_{Am}(x) = \min(\mu_{A1}(x), \mu_{A2}(x), \ldots.., \mu_{An}(x))$
- Now the compound rule can be rewritten as
- IF X is Am THEN Y is Bm

Multiple Disjunctive antecedents

- Example-IF X is A1 Or X is A2Or …, An THEN Y is Bm
- Decomposition
- Assume a fuzzy set

$$Am = A_1 \cup A_2 \cup \ldots . \cup An$$

- Expressed using the membership function
- $\mu_{Am}(x) = \max(\mu_{A1}(x), \mu_{A2}(x), \ldots.., \mu_{An}(x))$
- Now the compound rule can be rewritten as
- IF X is Am THEN Y is Bm

STATEMENTS WITH ELSE

- Example-IF A1 THEN (B1 ELSE B2)
- Can be decomposed as
- IF A1 THEN B1
- OR
- IF NOT A1 THEN B2

STATEMENTS WITH UNLESS

- Example-IF A1 (THEN B1) UNLESS A2
- Can be decomposed as
- IF A1 THEN B1
- OR
- IF A2 THEN NOT B1

STATEMENTS WITH ELSE IF

- Example-IF A1 THEN (B1) ELSE IF A2 THEN (B2)
- Can be decomposed as
- IF A1 THEN B1
- OR
- IF NOT A1 AND IF A2 THEN B2

STATEMENTS WITH NESTED-IF-THEN

- Example-IF A1 THEN [IF A2 THEN (B1)]
- Can be decomposed as
- IF A1 AND A2 THEN B1

Aggregation of Fuzzy Rules

- Rule based system involves more than one rule
- Aggregation- obtaining the overall consequent
- Two types of systems
- Conjunctive system-rules to be jointly satisfied
  - Y= y1 and y2 and … and yn
  - $\mu_{Am}(x) = \min(\mu_{A1}(x), \mu_{A2}(x), \ldots.., \mu_{An}(x)$
- Disjunctive system-any of the rules to be satisfied
  - Y= y1 or y2 or … or yn
  - $\mu_{Am}(x) = \max(\mu_{A1}(x), \mu_{A2}(x), \ldots.., \mu_{An}(x)$

Thank You